5/1/08 Lecture 10 outline

• Continue from last time: $|\psi\rangle$ is a quantum state, which can be written in x-basis as $\psi(x) = \langle x|\psi\rangle$ or in p-basis as $\phi(p) = \langle p|\psi\rangle$. Either gives a complete, orthogonal basis

$$\int dx|x\rangle\langle x| = 1, \qquad \langle x'|x\rangle = \delta(x - x')$$

$$\int dp|p\rangle\langle p| = 1, \qquad \langle p'|p\rangle = \delta(p - p')$$

and the change of basis is via $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$, which leads to the formulae of Fourier transforms. Continue to relate this to what we saw before about Fourier transforms and computing e.g. $\langle f(x)\rangle$ and $\langle F(p)\rangle$ in position or momentum space. The wavefunction in position space is $\psi(x) = \langle x|\psi\rangle$.

• If measuring observable to operator A, write $|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle$ (using completeness). The probability to measure $A = a_i$ in this state is then $|\langle a_i|\psi\rangle|^2$. Immediately after the measurement, the wavefunction collapses, $|\psi\rangle \to |a_i\rangle$. If operators A and B commute, they can be simultaneously diagonalized, and measurements of A and B don't interfere with each other. If they don't commute, they can't be simultaneously measured: measuring one alters the state of the other.

• Aside: consider the Schwartz inequality with $|v\rangle = (A - \langle A\rangle)|\psi\rangle$ and $|w\rangle = (B - \langle B\rangle)\psi\rangle$. It the follows, for A and B Hermitian, that

$$\langle (A - \langle A\rangle)^2\rangle\langle (B - \langle B\rangle)^2\rangle \geq |\langle\psi|(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle|^2.$$

Writing $(A - \langle A\rangle)(B - \langle B\rangle) = \frac{1}{2}[A, B] + \frac{1}{2}\{A - \langle A\rangle, B - \langle B\rangle\}$, it follows that

$$\Delta A\Delta B \geq \frac{1}{2}|\langle [A, B]\rangle|.$$

• Time evolution: In classical physics, any phase space variable $u(p_i, q_i, t)$ has $\frac{du}{dt} = \frac{\partial u}{\partial t} + \{u, H\}_{P.B.}$. We go from C.M. to Q.M. by replacing $\{A, B\} \to (1/i\hbar)[A, B]$, and classical relations with Q.M. expectation values. So in Q.M. we have

$$\frac{d}{dt}\langle\psi|u|\psi\rangle = \langle\psi|\frac{\partial u}{\partial t}|\psi\rangle + \frac{1}{i\hbar}\langle\psi|[u, H]|\psi\rangle.$$

In the Schrodinger picture, this can be understood as coming from the time-evolution of the state-ket, according to the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi\rangle.$$

Corresponds to the statement that the Hamiltonian is the generator of time translations.

• Energy eigenstates, a.k.a. eigenkets of the Hamilton, have a very simple time evolution. If $H|E\rangle = E|E\rangle$, then the time evolution is just by the phase factor $|E\rangle \to e^{-iEt/\hbar}|E\rangle$, which fits with $E = \hbar\omega$. Energy eigensates form a complete basis, so

$$|\psi(t=0)\rangle = \sum \int |E\rangle\langle E|\psi\rangle$$

and the time evolution is to

$$\psi(t)\rangle = \sum \int e^{-iEt/\hbar}|E\rangle\langle E|\psi\rangle.$$

• Free particle. $H = p^2/2m$. Momentum is conserved, because $[H, p] = 0$, and the energy eigenstates are simple the momentum eigenstates, $H|p\rangle = E(p)|p\rangle$, with $E(p) = p^2/2m$. An energy eigenstate can be a superposition of momentum eigenstates, $|E\rangle = c_1|p = \sqrt{2mE}\rangle + c_2|p = -\sqrt{2mE}\rangle$. Measuring the energy leaves the momentum sign degenerate, so the state could still be in a quantum superposition of moving left and right. Measuring the momentum would collapse the state to one or the other. The momentum eigenstates form a complete basis, so any wavefunction can be written as

$$|\psi\rangle = \int_{-\infty}^{\infty} dp|p\rangle\langle p|\psi\rangle$$

and the time evolution is given by

$$|\psi(t)\rangle = \int_{-\infty}^{\infty} dp\ e^{-ip^2t/2m\hbar}|p\rangle\langle p|\psi(t=0)\rangle.$$

Recall example of Gaussian wavepacket:

$$\langle x|\psi(t=0)\rangle = \frac{1}{(\pi\Delta^2)^{1/4}} e^{ip_0x/\hbar} e^{-x^2/2\Delta^2}$$

has $\langle X\rangle = 0$, $\langle P\rangle = p_0$, $\Delta X = \Delta/\sqrt{2}$, and $\Delta P = \hbar/\Delta\sqrt{2}$. Time evolution, using the above, gives a state with $\langle X\rangle = \langle P\rangle t/m$, and $\langle P\rangle = p_0$ and $\Delta X(t) = \frac{\Delta}{\sqrt{2}}\left(1 + \frac{\hbar^2t^2}{m^2\Delta^4}\right)^{1/2}$. This is the spreading of the wavepacket discussed before.