130a Homework 3, due 4/24

1a. Prove (in momentum space) that $\langle x^n \rangle = 0$, for all odd n, whenever $\phi(p)$ is real.

1b. Show that, if $\psi(x)$ has mean momentum $\langle p \rangle = \bar{p}$, then $e^{ip_0 x/\hbar}\psi(x)$ has mean momentum $\langle p \rangle = \bar{p} + p_0$.

2. Suppose that

$$\phi(p) = \begin{cases} A & \text{for } p_0 - b < p < p_0 + b \\ 0 & \text{otherwise,} \end{cases}$$

where A and b and p_0 are constants.

a. Solve for A (in terms of b) for the wavefunction to be properly normalized.

b. Compute $\langle p \rangle$, $\langle p^2 \rangle$ and $\Delta p \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, in terms of p_0 and the constant b.

c. Compute $\psi(x)$.

3. Suppose that $\psi(x) = u_n(x)$ given by

$$u_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} & \text{if } 0 \leq x \leq a \\ 0 & \text{otherwise,} \end{cases}$$

where n is an integer.

a. Compute $\langle x \rangle$ and $\langle x^2 \rangle$ and Δx.

b. Compute $\langle p \rangle$ and $\langle p^2 \rangle$ and Δp.

4. Fourier's theorem states that any function $\psi(x)$, defined for $0 \leq x \leq a$, which satisfies $\psi(0) = \psi(a) = 0$, can be written as a superposition of the u_n defined above:

$$\psi(x) = \sum_{n=1}^{\infty} A_n u_n(x),$$

where

$$A_n = \int_0^a dx\ u_n(x)^* \psi(x).$$

Suppose that

$$\psi(x) = \begin{cases} \sqrt{\frac{4}{a}} \sin \frac{2\pi x}{a} & \text{for } 0 \leq x \leq \frac{1}{2}a \\ 0 & \text{for } \frac{1}{2}a \leq x \leq a. \end{cases}$$

Compute the coefficients A_1 and A_2 in the above expansion.

5. Consider the Poisson bracket for a single coordinate x, and conjugate momentum p. Recall the definition

$$\{A, B\} \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}.$$

Recall we take x and p to be two independent coordinates (this is called phase space), and $A = A(x, p)$ and $B = B(x, p)$ are any functions of these coordinates. It is obvious from the definition above that $\{A, B\} = -\{B, A\}$. Verify the following other identities, for general functions A, B, and C of the phase space coordinates x and p:

a. $\{AB, C\} = A\{B, C\} + \{A, C\}B$.

b. $\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$.

Now recall the definition of the commutator of two operators, $[A, B] \equiv AB - BA$. It is obvious from the definition that $[A, B] = -[B, A]$. Using the definition, verify that

c $[AB, C] = A[B, C] + [A, C]B$.

d. $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$.

6. Exercises 1.1.5 and 1.3.4 in Shankar.

7. Exercise 1.8.3 in Shankar.