6/5/07 Lecture 18 outline

• Decoupled systems, e.g. $H = H_1 + H_2$. Energy eigenstates and eigenvalues. Relate to separation of variables.

• Consider spherically symmetric $H = \frac{1}{2m}\vec{p}^2 + V(r)$. Since $[H, L_i] = 0$, we can find simultaneous eigenstates $|E, \ell, m\rangle$ of H, L^2, and L_z. Indeed, note that

$$H = \frac{p_r^2}{2\mu} + \frac{L^2}{2\mu r^2} + V(r)$$

where in position space the first terms correspond to

$$\vec{p}^2 \to -\hbar^2\,\nabla^2 = -\hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}\right) \equiv p_r^2 + \frac{L^2}{r^2}.$$

It is common, also in classical mechanics, to note that this looks like a 1d problem now, with $V_{eff}(r) = V(r) + (L^2/2\mu r^2)$. Here μ is the mass, not to be mistaken for the m integer appearing in $|\ell, m\rangle$. The energy eigenvalue equation then becomes

$$\left(\frac{p_r^2}{2\mu} + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} + V(r)\right)|E, \ell, m\rangle = E|E, \ell, m\rangle.$$

In position space, write this equation and discuss solution by separation of variables. We then have $\psi_{E,\ell,m}(r,\theta,\phi) = \langle r,\theta,\phi|E,\ell,m\rangle = R_{n,\ell}(r)Y_{\ell,m}(\theta,\phi)$, where

$$\left(-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} + V(r)\right)R_{n,\ell}(r) = E_{n,\ell}R_{n,\ell}(r),$$

where n labels the solutions of this equation. The energy is quantized, via n, because we're considering bound states of the potential. Note that the energy eigenvalues $E_{n,\ell}$ don't depend on the L_z eigenvalue m; this is as expected from the spherical symmetry, which implies that $[H, L_\pm] = 0$. The derivative terms become a little simpler if we define $R_{n,\ell}(r) = U_{n,\ell}(r)/r$:

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V(r) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)U_{n,\ell}(r) = E_{n,\ell}U_{n,\ell}(r).$$

• Consider the limit $r \to 0$. If $\ell \neq 0$, and if $V(r)$ is less singular than r^{-2} as $r \to 0$, then the terms $V(r)$ and the term E become negligible as $r \to 0$, and we have $U_\ell''(r) \approx \ell(\ell+1)r^{-2}U_\ell(r)$, i.e. for $r \to 0$, we have $R_\ell(r) = U_\ell(r)/r \to A_\ell r^\ell + B_\ell r^{-(\ell+1)}$, as is familiar from solving the Laplace eqn. in spherical coordinates. We need to set $B_\ell = 0$ to have a regular solution at $r = 0$. Now consider $r \to \infty$ and suppose that $V \to 0$ in this limit. Then the $E < 0$ bound state wavefunctions behave for $r \to \infty$ as $U(r) \sim e^{-\kappa r}$ in this limit, with $\kappa = \sqrt{2\mu|E|/\hbar^2}$.