5/29/07 Lecture 16 outline

• Note that different components of the angular momentum don't commute! Angular momentum commutation relations $[L_x, L_y] = i\hbar L_z$ and cyclic permutations. Implies that we can't measure different components simultaneously.

• Convention is to diagonalize L_z. Show that $[L^2, L_z] = 0$, so can also diagonalize L^2. Let's call their eigenkets $|\alpha, \beta\rangle$, where $L^2|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle$, and $L_z|\alpha, \beta\rangle = \beta|\alpha, \beta\rangle$. If we measure L^2 and L_z in an experiment, the eigenvalues α and β, respectively, are the only possible outcomes. The above angular momentum commutation relations constrain α and β, and implies that they are quantized.

• Classically, α could take any non-negative value, $\alpha \geq 0$. And classically, β can take any value, in the range $-\sqrt{\alpha} \leq \beta \leq \sqrt{\alpha}$, since the z component of the vector $\vec{L}$ can't exceed $\pm|\vec{L}|$. We can prove these same inequalities in Q.M., and actually make them even stronger. To prove the inequalities, recall that for any ket $|\psi\rangle$, the bra-ket $\langle\psi|\psi\rangle \equiv |||\psi\rangle||^2 \geq 0$, with $|||\psi\rangle||^2 \geq 0$ if and only if the ket vanishes, $|\psi\rangle = 0$. Moreover, for any operator A,

$$\langle A^\dagger A\rangle \equiv \langle\psi|A^\dagger A|\psi\rangle \equiv ||A|\psi\rangle||^2 \geq 0,$$

again with equality iff $A|\psi\rangle = 0$. In QM we have $L_a^\dagger = L_a$, as is the case for all physical observables. So we see from the above that $\langle L_x^2\rangle \geq 0$, and $\langle L_y^2\rangle \geq 0$ and $\langle L_z^2\rangle \geq 0$. In particular, in the state $|\psi\rangle = |\alpha, \beta\rangle$, we have $\langle L^2\rangle = \alpha \geq 0$, and also $\langle L_x^2 + L_y^2\rangle = \langle L^2 - L_z^2\rangle = \alpha - \beta^2 \geq 0$. So the classical inequalities are satisfied. But the classical inequality $\alpha - \beta^2 \geq 0$ is too weak in general: note that $\alpha - \beta^2 = \langle L_x^2\rangle + \langle L_y^2\rangle$, and we can't set both terms on the right hand side to zero, in general, because of the $[L_x, L_y] = i\hbar L_z$, which implies an uncertainty principle-like inequality for the product $\Delta L_x \Delta L_y$, saying that both can't vanish. To show this, and more, let's introduce the $L_\pm$ operators.

• Raising and lowering operators (analogous to creation and annihilation operators in SHO): $L_\pm \equiv L_x \pm iL_y$, satisfy $[L_z, L_\pm] = \pm\hbar L_\pm$. It then follows that $L_\pm|\alpha\beta\rangle \sim |\alpha, \beta \pm \hbar\rangle$. Note that $[L^2, L_\pm] = 0$, so $L_\pm$ raise and lower the L_z component of angular momentum, but leave the magnitude L^2 of the angular momentum vector unchanged. It's like they rotate the $\vec{L}$ vector to point more, or less, along the $\hat{z}$ axis.

• Note $L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z$, and that $L_\pm^\dagger = L_\mp$. So $\langle L_\pm L_\mp\rangle \geq 0$ and $\langle L_\pm L_\mp\rangle \geq 0$ in any state. In particular, in the state $|\alpha\beta\rangle$ we have

$$\langle L_+L_-\rangle = \alpha - \beta^2 + \hbar\beta \geq 0 \qquad \text{and} \qquad \langle L_-L_+\rangle = \alpha - \beta^2 - \hbar\beta \geq 0,$$

where we've fixed the normalization by $\langle\alpha,\beta|\alpha,\beta\rangle = 1$. Note that this also determines the normalization in $L_\pm|\alpha\beta\rangle \sim |\alpha,\beta\pm\hbar\rangle$:

$$L_\pm|\alpha,\beta\rangle = \sqrt{\alpha-\beta^2\mp\hbar\beta}|\alpha,\beta\pm\hbar\rangle.$$

• But we saw that we can raise and lower β by acting on $|\alpha,\beta\rangle$ with $L_\pm$, which leaves α unchanged but takes $\beta\to\beta\pm\hbar$. If α and β were general numbers, we'd then violate the above inequalities. The only way to avoid this is if there is a β_{max}, such that $L_+|\alpha,\beta_{max}\rangle = 0$, and a β_{min} such that $L_-|\alpha,\beta_{min}\rangle = 0$. It follows from the above then that $\alpha = \beta_{max}^2 + \beta_{max}\hbar = \beta_{min}^2 - \beta_{min}\hbar$. So $\beta_{min} = -\beta_{max}$. Moreover, must have that $L_-^N|\alpha,\beta_{max}\rangle \sim |\alpha,\beta_{max}-N\rangle$ must eventually vanish, so there is some integer N such that $\beta_{max}-N = \beta_{min}$, i.e. $2\beta_{max} = N$. So β_{max} can either be an integer or a half integer.

• For orbital angular momentum, $\beta_{max}\equiv\ell$ is an integer. Nature also use the half-integer possibility, in the context of spin: fermions have half-integer total angular momentum, given by $\vec{J} = \vec{L}+\vec{S}$, where $\vec{L}$ is the orbital part and $\vec{S}$ is the spin part. Ignore $\vec{S}$ for now, discuss it later.

• Instead of labeling the kets by α and β, label by ℓ and m, where $\alpha = \hbar^2\ell(\ell+1)$ and $\beta = \hbar m$, and m runs from ℓ to $-\ell$, in integer steps (so there are $2\ell+1$ values of m):

$$L^2|\ell,m\rangle = \hbar^2\ell(\ell+1)|\ell,m\rangle \qquad L_z|\ell,m\rangle = m\hbar|\ell,m\rangle$$

and

$$L_\pm|\ell,m\rangle = \hbar\sqrt{\ell(\ell+1)-m^2\mp m}|\ell,m\pm 1\rangle.$$

• The $|\ell,m\rangle$ form a complete, orthonormal basis:

$$\langle\ell',m'|\ell,m\rangle = \delta_{\ell,\ell'}\delta_{m,m'} \qquad \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}|\ell,m\rangle\langle\ell,m| = \mathbf{1}.$$

• Next time: Consider these kets in position space. Use spherical coordinates. The $|\ell,m\rangle$ states are independent of the radial coordinate, r; they depend only on θ and ϕ. To see why, write $\vec{L} = \vec{x}\times\vec{p}$ in position space, by replacing $\vec{p}\to -i\hbar\nabla$. Converting to spherical coordinates, get

$$L_z \to -i\hbar\frac{\partial}{\partial\phi} \qquad L_\pm \to \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

and

$$L^2 \to -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right].$$