

5/22/07 Lecture 14 outline

- The harmonic oscillator, using a and a^\dagger , with $[a, a^\dagger] = 1$ and $H_{sho} = \hbar\omega(a^\dagger a + \frac{1}{2})$.
- Show $a^\dagger a|\lambda\rangle = \lambda|\lambda\rangle$, with $\lambda \geq 0$.
- Show $a|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle$, and $a^\dagger|\lambda\rangle = \sqrt{\lambda + 1}|\lambda + 1\rangle$. Note that $a^k|\lambda\rangle \sim |\lambda - k\rangle$.
- Must then have a state annihilated by a , i.e. $\lambda = n$ and $a|0\rangle = 0$.
- Thus $H_{sho}|n\rangle = E_n|n\rangle$ with $E_n = (n + \frac{1}{2})\hbar\omega$. Moreover, $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$.
- Using $a = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i(2m\omega\hbar)^{-1/2}\hat{p}$, convert the above to position space. E.g. $\psi_0(x) = (m\omega/\pi\hbar)^{1/4} \exp(-m\omega x^2/2\hbar)$. Defining $y \equiv \sqrt{m\omega/\hbar}x$, get

$$\psi_n(x) = (2^n n!)^{-1/2} (m\omega/\pi\hbar)^{1/4} (y - \frac{d}{dy})^n e^{-y^2/2}.$$

But it's almost always better to work with the basis independent bras, kets, and the operators a and a^\dagger .

• **Summary of everything you need to know, to do the homework:** The states $|n\rangle$, with $n = 0, 1, 2, \dots$, form an orthonormal, complete basis:

$$\langle n|m\rangle = \delta_{nm} \quad \text{and} \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1_{op}.$$

The annihilation and operators act on them as

$$a|n\rangle = \sqrt{n}|n - 1\rangle \quad a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle.$$

To compute things like $\langle n|\hat{x}^k|m\rangle$ etc, it is useful to express \hat{x} and \hat{p} in terms of a and a^\dagger , using

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \quad \hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}} (a - a^\dagger).$$

• Example: Evaluate Δx and Δp in the state $|1\rangle$. Use $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$ and $\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2$. Note that $\langle 1|a^k|1\rangle = 0$ for all $k > 0$. Thus $\langle x\rangle = \langle p\rangle = 0$ in the state $|1\rangle$. To compute $\langle x^2\rangle$, use $\hat{x}^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2$, and note that only the 2nd and 3rd terms in $(a + a^\dagger)^2 = a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}$ give a non-zero contribution when sandwiched between $\langle 1|$ and $|1\rangle$. To compute these, use the above expressions for how a and a^\dagger act on $|n\rangle$ to get $a^\dagger a|1\rangle = |1\rangle$, and $aa^\dagger|1\rangle = 2|1\rangle$, so $\langle x^2\rangle = \frac{\hbar}{2m\omega}(1 + 2)$. Likewise, get $\langle p^2\rangle = -\frac{m\omega\hbar}{2}(-1 - 2)$. Note that $\Delta x\Delta p = \frac{3}{2}\hbar$, so the uncertainty principle inequality is comfortably satisfied.