5/17/07 Lecture 13 outline

• In quantum mechanics, we replace physical observables, like x, p, E, etc. with Hermitian operators. The observed quantities are the eigenvalues. Write e.g.

$$\widehat{x}|x\rangle = x|x\rangle, \qquad \widehat{p}|p\rangle = p|p\rangle, \qquad H|E\rangle = E|E\rangle.$$

These operators generally act in an infinite dimensional space, the Hilbert space, but this generally doesn't complicate things much (from a physicist's perspective).

The x and p eigenvalues are continuous. E can be continuous (e.g. free particle), discrete (e.g. particle in a box with $V = \infty$ outside, or more generally when $V \to \infty$ for $|x| \to infty$), or a combination of continuous and discrete (e.g. for the particle in a finite potential well).

• As we have discussed, the operators $\widehat{x}$ and $\widehat{p}$ satisfy $[\widehat{x}, \widehat{p}] = i\hbar$. The fact that they don't commute means that they don't have simultaneous eigenvectors, they can't be simultaneously diagonalized. Nevertheless, there is a close relation between them. For any ket χ, we have

$$\langle x|\widehat{p}|\chi\rangle = -i\hbar \frac{d}{dx}\langle x|\chi\rangle \qquad \text{and} \langle p|\widehat{x}|\chi\rangle = i\hbar \frac{d}{dp}\langle p|\chi\rangle.$$

These relations are consistent with $[\widehat{x}, \widehat{p}] = i\hbar$.

Their separate eigenkets satisfy $\langle x'|x\rangle = \delta(x - x')$, and $\langle p'|p\rangle = \delta(p - p')$. The completeness relations are

$$\int_{-\infty}^{\infty} dx|x\rangle\langle x| = 1_{op} \qquad \int_{-\infty}^{\infty} dp|p\rangle\langle p| = 1_{op}.$$

The relation between these bases is $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$. Note that this satisfies $\langle x|p|p\rangle = (-i\hbar\frac{d}{dx})\langle x|p\rangle = p\langle x|p\rangle$.

The wavefunction of a QM system is represented by an abstract vector in the Hilbert space, $|\psi\rangle$. The wavefunction in position space is $\psi(x) = \langle x|\psi\rangle$. The wavefunction in momentum space is $\phi(p) = \langle p|\psi\rangle$. Explain the meaning of the Fourier transform between them, with $\langle x|p\rangle$.

• **Measurement (this is a key point!):** If measuring observable to operator A, write $|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle$ (using completeness). The probability to measure $A = a_i$ in this state is then $|\langle a_i|\psi\rangle|^2$. Immediately after the measurement, the wavefunction collapses, $|\psi\rangle \to |a_i\rangle$. If operators A and B commute, they can be simultaneously diagonalized. Discuss measurement and operators which do, or do not, commute.

• The Schrodinger equation in this notation is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi\rangle.$$

The way we wrote it before was in the x basis. Discuss it in other bases.

• E.g. for the particle in the infinite potential well, we have energy eigenstates $|n\rangle$, $n = 1, 2, \ldots$, with $H|n\rangle = E_n|n\rangle$, for $E_n = n^2\pi^2\hbar^2/2mL^2$. These energy eigenstates satisfy the usual relations for a complete, orthonormal basis:

$$\langle n|m\rangle = \delta_{nm} \qquad \sum_{n=1}^{\infty} |n\rangle\langle n| = 1.$$

The relation between the $|n\rangle$ and the $|x\rangle$ basis is

$$\langle x|n\rangle = u_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin(n\pi x/L) & \text{for } 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}.$$