5/15/07 Lecture 12 outline

• Recall operators A map kets to kets, e.g. $A|v\rangle = |Av\rangle$. Taking the adjoint, $\langle Av| = \langle v|A^\dagger$.

• Ae can write a general operator A as

$$A = \sum_{ij=1}^{K} |e_i\rangle\langle e_i|A|e_j\rangle\langle e_j|,$$

where $\langle e_i|A|e_j\rangle = A_{ij}$ are the matrix elements, and the other pieces are the basis elements for $K \times K$ matrices. Note that we can get the above by using the completeness relation twice.

The adjoint operation then acts as

$$A^\dagger = \sum_{ij=1}^{K} |e_j\rangle\langle e_j|A^\dagger|e_i\rangle\langle e_i|,$$

The order is reversed, and bras and kets are exchanged. Note that $\langle e_j|A^\dagger|e_i\rangle = \langle e_i|A|e_j\rangle^*$.

• The equation for an eigenvector and eigenvalue is $A|a_i\rangle = a_i|a_i\rangle$, where the eigenvector is labeled by the eigenvalue a_i, for $i = 1 \ldots K$.

• Suppose A is Hermitian, $A^\dagger = A$. Then $\langle a_i|A|a_i\rangle^* = a_i\langle a_i|a_i\rangle = \langle a_i|A^\dagger|a_i\rangle = a_i\langle a_i|a_i\rangle$, from which it follows that $a_i = a_i^*$; the eigenvalues of Hermitian operators are real.

Also, using $A - A^\dagger = 0$, get $0 = \langle a_i(A - A^\dagger)|a_j\rangle = (a_j - a_i)\langle a_i|a_j\rangle$, so $a_i \neq a_j$ implies that $\langle a_i|a_j\rangle = 0$; eigenvectors with different eigenvectors are orthogonal.

We can use the eigenvectors of a Hermitian operator to form a (complete) basis, with $\langle a_i|a_j\rangle = \delta_{ij}$ and $\sum_i |a_i\rangle\langle a_i| = 1$ (if there are many eigenvectors with the same eigenvalue, all have to be included in these sums). In this basis, $A = \sum_i a_i|a_i\rangle\langle a_i|$ corresponds to a diagonal matrix. This is the statement that A can be diagonalized by a similarity transformation, given by the matrix of eigenvectors.

• If $[A, B] = 0$, then A and B can be simultaneously diagonalized. If $[A, B] \neq 0$, then they can not.

• Define expectation values in state $|\psi\rangle$ by $\langle A\rangle \equiv \langle\psi|A|\psi\rangle$. If A is Hermitian, then $\langle A\rangle$ is real. Note also that $\langle A\rangle = \sum_i a_i|\langle a_i|\psi\rangle|^2$.

• Consider the Schwartz inequality with $|v\rangle = (A - \langle A\rangle)|\psi\rangle$ and $|w\rangle = (B - \langle B\rangle)\psi\rangle$. It the follows, for A and B Hermitian, that

$$\langle(A - \langle A\rangle)^2\rangle\langle(B - \langle B\rangle)^2\rangle \geq |\langle\psi|(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle|^2.$$

Writing $(A - \langle A \rangle)(B - \langle B \rangle) = \frac{1}{2}[A, B] + \frac{1}{2}\{A - \langle A \rangle, B - \langle B \rangle\}$, it follows that

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle|.$$

Now let's connect all this with what we've seen in quantum mechanics.

• In quantum mechanics, we replace physical observables, like x, p, E, etc. with Hermitian operators. The observed quantities are the eigenvalues. Write e.g.

$$\widehat{x}|x\rangle = x|x\rangle, \qquad \widehat{p}|p\rangle = p|p\rangle, \qquad H|E\rangle = E|E\rangle.$$

These operators generally act in an infinite dimensional space, the Hilbert space, but this generally doesn't complicate things much (from a physicist's perspective).

• As we have discussed, the operators $\widehat{x}$ and $\widehat{p}$ satisfy $[x, p] = i\hbar$. The fact that they don't commute means that they don't have simultaneous eigenvectors, they can't be simultaneously diagonalized.

Their separate eigenkets satisfy $\langle x'|x\rangle = \delta(x - x')$, and $\langle p'|p\rangle = \delta(p - p')$. The completeness relations are

$$\int_{-\infty}^{\infty} dx|x\rangle\langle x| = 1_{op} \qquad \int_{-\infty}^{\infty} dp|p\rangle\langle p| = 1_{op}.$$

The relation between these bases is $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$. Note that this satisfies $\langle x|p|p\rangle = (-i\hbar\frac{d}{dx})\langle x|p\rangle = p\langle x|p\rangle$.

The wavefunction of a QM system is represented by an abstract vector in the Hilbert space, $|\psi\rangle$. The wavefunction in position space is $\psi(x) = \langle x|\psi\rangle$. The wavefunction in momentum space is $\phi(p) = \langle p|\psi\rangle$. Explain the meaning of the Fourier transform between them, with $\langle x|p\rangle$.

• **Measurement (this is a key point!):** If measuring observable to operator A, write $|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle$ (using completeness). The probability to measure $A = a_i$ in this state is then $|\langle a_i|\psi\rangle|^2$. Immediately after the measurement, the wavefunction collapses, $|\psi\rangle \to |a_i\rangle$. If operators A and B commute, they can be simultaneously diagonalized. Discuss measurement and operators which do, or do not, commute.

• The Schrodinger equation in this notation is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi\rangle.$$

The way we wrote it before was in the x basis. Discuss it in other bases.