5/10/07 Homework 5 Due May 17, 2007.

1. Gas. 4.8
2. Gas. 4.10a. There is a small typo in the problem: the "a" on the 3rd line of the given $V(x)$ should not be in italics. Hint: write the form of the wavefunction for $x > a$, in terms of the energy E_B.
3. Gas. 4.14
4. For a finite dimensional, complex vector space, prove the Schwartz inequality,

$$\langle v|v\rangle\langle w|w\rangle \geq |\langle v|w\rangle|^2.$$

   To do this, use the fact that $\langle z|z\rangle \geq 0$ for $|z\rangle = |v\rangle + \lambda|w\rangle$, where λ is a free parameter, and find the value of λ which minimizes $\langle z|z\rangle$.

5. Recall that an operator A is hermitian if $A = A^\dagger$.
   (a) Show that. if A and B are Hermitian, then so is $(A+B)^n$.
   (b) Show that, if A and B are Hermitian, then so is $i[A, B]$.
   (c) Gas. 6.1b
   (d) Show that $A^\dagger A$ is Hermitian, even if A is not.
6. Consider a $K = 2$ dimensional, complex vector space, with orthonormal basis kets $|e_i\rangle$, for $i = 1, 2$. (For example, you can take $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$). Consider the operator $A = e^{i\phi}|e_1\rangle\langle e_2| + e^{-i\phi}|e_2\rangle\langle e_1|$, where ϕ is an arbitrary real number.
   (a) Verify that $A^\dagger = A$.
   (b) Compute the eigenvalues λ_1 and λ_2 of A, and verify that they are real.

(c) Find the corresponding eigen-kets $|\lambda_1 >$ and $|\lambda_2 >$, as linear combinations of the $|e_i\rangle$. Verify that the $|\lambda_i\rangle$ are orthogonal, and normalize them so that they are orthonormal, $< \lambda_i|\lambda_j >= \delta_{ij}$.

(d) Using the above results, explicitly verify that $A = \sum_{i=1}^{2} \lambda_i |\lambda_i\rangle\langle\lambda_i|$.

(e) Compute the expectation value of the operator A, defined to be $\langle A\rangle \equiv \langle\psi|A|\psi\rangle$, in the state $|\psi\rangle >\equiv a|e_1\rangle + b|e_2\rangle$, where a and b are arbitrary *complex* numbers. (Your answer for $\langle A\rangle$ should be real, since that's the case for any Hermitian operator, in any state.)

7. Using $[\hat{x}, \hat{p}] = i\hbar$, verify
   (a) that $\hat{a} \equiv \sqrt{\frac{\alpha}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\alpha\hbar}}\hat{p}$ satisfies $[\hat{a}, \hat{a}^\dagger] = 1$, for arbitrary real parameter α.

(b) Compute $a^\dagger a$ and show that it has three terms, one proportional to $\hat{x}^2$, one to $\hat{p}^2$, and a constant.