4/20/07 Homework 3. Due April 26, 2007.

1. Prove (in momentum space) that $\langle x^n \rangle = 0$, for all odd n, whenever $\phi(p)$ is real.

2. Suppose that

$$\phi(p) = \begin{cases} A & \text{for } p_0 - b < p < p_0 + b \\ 0 & \text{otherwise,} \end{cases}$$

where A and b and p_0 are constants.

a. Solve for A (in terms of b) for the wavefunction to be properly normalized.

b. Compute $\langle p \rangle$, $\langle p^2 \rangle$ and $\Delta p \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, in terms of p_0 and the constant b.

c. Compute $\psi(x)$.

3. Consider the Poisson bracket for a single coordinate x, and conjugate momentum p. Recall the definition

$$\{A, B\} \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}.$$

Recall we take x and p to be two independent coordinates (this is called phase space), and $A = A(x,p)$ and $B = B(x,p)$ are any functions of these coordinates. It is obvious from the definition above that $\{A,B\} = -\{B,A\}$. Verify the following other identities, for general functions A, B, and C of the phase space coordinates x and p:

a. $\{AB, C\} = A\{B,C\} + \{A,C\}B$.

b. $\{A, \{B,C\}\} + \{C, \{A,B\}\} + \{B, \{C,A\}\} = 0$.

4. Recall the definition of the commutator of two operators, $[A,B] \equiv AB - BA$. It is obvious from the definition that $[A,B] = -[B,A]$. Using the definition, verify that

a $[AB, C] = A[B,C] + [A,C]B$.

b. $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$.

5. Consider the wavefunction given in class for a free particle moving with momentum p_0 (and spreading).

$$\psi(x,t) = \left(\frac{1}{\sqrt{2\pi(\sigma + it\hbar/2m\sigma)}}\right)^{1/2} e^{ip_0(x - p_0 t/2m)/\hbar} \exp(-(x - p_0 t/m)^2/4(\sigma + it\hbar/2m\sigma)^2).$$

Compute the probability density $\rho(x,t) = |\psi(x,t)|^2$ and current density $J(x,t) = \frac{\hbar}{2im}\left(\psi^* \frac{\partial}{\partial x}\psi - \frac{\partial \psi^*}{\partial x}\psi\right)$. (Make sure to express your results so that these are purely real - there should be no i's in your final expressions for ρ and J.)

This next part is a pain, so it is now OPTIONAL (extra credit for trying): Verify that your answers satisfy the general conservation law $\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$.

6. Gas. 3.3. (The definition of the operators there is as given in problem 3.1.)

7. Gas 3.4

8. Gas. 3.14b. Work it out in position space, rather than momentum space.

9. Gas. 3.15. His term "flux" means the same thing as the probability current density J defined above.

10. Gas. 3.16.

11. Fourier's theorem states that any function $\psi(x)$, defined for $0 \leq x \leq L$, which satisfies $\psi(0) = \psi(L) = 0$, can be written as

$$\psi(x) = \sum_{n=1}^{\infty} A_n u_n(x) \qquad u_n \equiv \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L},$$

where

$$A_n = \int_0^L dx\; u_n(x)^* \psi(x).$$

Suppose that

$$\psi(x) = \begin{cases} \sqrt{\frac{4}{L}} \sin \frac{2\pi x}{L} & \text{for } 0 \leq x \leq \frac{1}{2}L \\ 0 & \text{for } \frac{1}{2}L \leq x \leq L. \end{cases}$$

Compute the coefficients A_1 and A_2 in the above expansion.