10/23/19 Lecture outline

• Last time: amplitudes in toy model of real mesons ϕ of mass μ and complex nucleons of mass m, with $H_{int} = -g\phi\bar{\psi}\psi$. Aside, quantize the nucleons as usual gives $[\psi(\vec{x},t), \dot{\psi}^\dagger(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$. Consider $N+N\to N+N$, to $\mathcal{O}(g^2)$. The initial and final states are

$$|i\rangle = b^\dagger(p_1)b^\dagger(p_2)|0\rangle, \qquad \langle f| = \langle 0|b(p_1')b(p_2').$$

The term that contributes to scattering at $\mathcal{O}(g^2)$ is **(don't forget the time ordering!)**

$$T\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 \phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S-1$ thus involves

$$\langle p_1'p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1p_2\rangle = \langle p_1'p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1,p_2\rangle.$$

$$= \left(e^{i(p_1'x_1+p_2'x_2)} + e^{i(p_1'x_2+p_2'x_1)}\right)\left(e^{-i(p_1x_1+p_2x_2)} + e^{-i(p_1x_2+p_2x_1)}\right).$$

The amplitude involves this times $D_F(x_1-x_2)$ (from the contraction), with the prefactor and integrals as above. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1-p_1')^2-\mu^2+i\epsilon} + \frac{1}{(p_1-p_2')^2-\mu^2+i\epsilon}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2').$$

Explicitly, in the CM frame, $p_1 = (\sqrt{p^2+m^2}, p\hat{e})$ and $p_2 = (\sqrt{p^2+m^2}, -p\hat{e})$, $p_1' = (\sqrt{p^2+m^2}, p\hat{e}')$, $p_2' = (\sqrt{p^2+m^2}, -p\hat{e}')$, where $\hat{e}\cdot\hat{e}' = \cos\theta$, and get

$$\mathcal{A} = g^2\left(\frac{1}{2p^2(1-\cos\theta)+\mu^2} + \frac{1}{2p^2(1+\cos\theta)+\mu^2}\right).$$

According to the above, $[\mathcal{A}(2\to 2)] = 0$ and the above is consistent with that. Good.

Note also that the amplitude is symmetric if we exchange $p_1^\mu \leftrightarrow p_2^\mu$ and likewise for the outgoing states. This fits with the fact that the N states are identical bosons, which follows from the fact that $[\psi(t,\vec{x}), \psi(t,\vec{y})] = 0$. As we'll discuss later, identical fermions instead have $\{\psi(t,\vec{x}), \psi(t,\vec{y})\} = 0$.

• Mandelstam variables for $p_1+p_2\to p_1'+p_2'$ scattering: $s=(p_1+p_2)^2$, $t=(p_1-p_1')^2$, $u=(p_1-p_2')^2$, with $s+t+u = m_1^2+m_2^2+m_{1'}^2+m_{2'}^2$. In CM, $s=4E^2$, $t=-2p^2(1-\cos\theta)$, and $u=-2p^2(1+\cos\theta)$.

• Recall how we got the above answer. We expand $\exp(-ig\int d^4x\mathcal{H})$ and compute the time ordered expectation values using Wick's theorems, with the contractions giving factors of $D_F(x_1-x_2)$. Doing this, we get a $\int d^4x$ for each factor of $-ig$ and a d^4k for each internal contraction. Draw a picture in position space. Let E be the number of external lines, i.e. the number of incoming + outgoing particles. (We saw last time that $[\mathcal{A}]=4-E$.) It is easier to think about everything in momentum space. Then the $\int d^4x$ for each vertex gives a $(2\pi)^4\delta^4(p_{total,\ in})$.

• Feynman rules! Each vertex gets $(-ig)(2\pi)^4\delta^4(p_{total\ in})$, each internal line gets $\int\frac{d^4k}{(2\pi)^4}D_F(k^2)$, where D_F is the propagator, e.g. $D_F(k^2)=\frac{i}{k^2-m^2+i\epsilon}$. Result is $\langle f|(S-1)|i\rangle$, so divide by $(2\pi)^4\delta^4(p_F-p_I)$ to get $i\mathcal{A}_{fi}$.

If the diagram has no loops, the momentum conserving delta functions fix all internal momenta in terms of the external ones. When the diagram has $L\neq 0$ loops, the procedure above yields integrals over the internal momenta of the loops. (Note that if a diagram has I internal lines and V vertices, then there are I momentum integrals, and V momentum conserving delta functions; one of these becomes overall momentum conservation, so there are $L=I-(V-1)$ momentum integrals left to do, and L is the number of loops in the diagram.) Any loop momentum integrals require renormalization, which we'll discuss later (next quarter), so for now we'll just consider "tree-level" contributions, associated with diagrams without loops, $L=0$.

• Scattering by ϕ exchange leads to an attractive Yukawa potential. This was Yukawa's original goal, to explain the attraction between nucleons. Indeed, the t-channel term in e.g. the above $N+N$ scattering amplitude gives, upon using $(p_1-p_1')^2-\mu^2=-(|\vec{p}_1-\vec{p}_1'|^2+\mu^2)$, and the Born approximation[1] in NRQM, $\mathcal{A}_{NR}=\int d^3\vec{r}e^{-i(\vec{p}'-\vec{p})\cdot\vec{r}}V(\vec{r})$, the attractive Yukawa potential

$$V(r)=\int\frac{d^3q}{(2\pi)^3}\frac{-(g/2m)^2e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2+\mu^2}=-\frac{(g/2m)^2}{4\pi r}e^{-\mu r}.$$

(The $1/(2m)^2$ is because we normalized the relativistic states with the extra factor of $2E\approx 2m$ as compared with standard nonrelativistic normalization[2]. This gives Yukawa's explanation of the attraction between nucleons, from meson exchange. The u-channel term is an exchange potential interaction, which exchanges the positions of the two identical particles in addition to giving a potential. For angular momentum ℓ in a partial-wave expansion, the exchange term differs from the direct one by a factor of $(-1)^\ell$.

[1] Max Born, in QM, or Lord Rayleigh classically: $\frac{d\sigma}{d\Omega}\sim|U(\vec{q})|^2$

[2] This is clear on dimensional grounds, since $[g]\sim m$. More generally, write $a(p)=\sqrt{2E}\widehat{a}(p)$ and $\mathcal{A}=\prod_i\sqrt{2E_i}\prod_f\sqrt{2E_f}\widehat{\mathcal{A}}$.