10/21/19 Lecture outline

• Last time: interaction picture. Write $H = H_0 + H_{int}$. We use H_0 to time evolve the operators, and H_{int} to time evolve the states:

$$i\frac{d}{dt}\mathcal{O}_I(t) = [\mathcal{O}_I, H_0], \qquad i\frac{d}{dt}|\psi(t)\rangle_I = H_{int}|\psi(t)\rangle_I.$$

$$|\psi(t)\rangle_I = e^{iH_0(q_S,p_S)t}|\psi(t)\rangle_S, \qquad \mathcal{O}_I = e^{iH_0t}\mathcal{O}_S e^{-iH_0t}$$

For example, we'll take H_0 to be the free Hamilton of KG fields, with only the mass terms included in the potential. Again, this is free because the EOM are linear, and we can solve for $\phi(x)$ by superposition. $H_I(t)$ is built from these free fields

$$\phi(\vec{x}, t) = e^{iH_0t}\phi_S(\vec{x})e^{-iH_0t}.$$

As before, upon quantization, the fields become superpositions of creation and annihilation operators. The states are all the various multiparticle states, coming from acting with the creation operators on the vacuum. Time evolution is via the interaction picture operator that satisfies

$$i\frac{d}{dt}U_I(t,t') = H_I(t)U_I(t,t').$$

Compute probabilities from squaring amplitudes, and amplitudes from $\langle f(t=+\infty)|i(t=-\infty)\rangle = \langle f|S|i\rangle = \langle f|U(\infty,-\infty)|i\rangle$. Naively, $U(t_f,t_i) = \exp(-\frac{i}{\hbar}\int_{t_i}^{t_f} H_{int}(t)dt)$, but have to be careful about H_{int} not commuting at different times. Get time ordering.

Dirac's / Dyson's formula:

$$U_I(t,t') = Te^{-i\int_{t'}^{t} dt'' H_I(t'')}.$$

Argue for it by iterating time intervals. To compute scattering S-matrices, a way to think about it (to be improved shortly) is to consider asymptotic in and out states, with the interaction turned off. Time evolve, with the interaction smoothly turned on and off in the middle (see Coleman notes for more details).

$$|\psi(t)\rangle = Te^{-i\int d^4x \mathcal{H}_I}|i\rangle.$$

Derive it by solving $i\frac{d}{dt}|\psi(t)\rangle = H_I(t)|\psi(t)\rangle$ iteratively:

$$|\psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\psi(t_1)\rangle$$

$$|\psi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t_1} dt_2 H_I(t_2)|\psi(t_2)\rangle$$

etc where $t_1 > t_2$, and then symmetrize in t_1 and t_2 etc., which is what the T time ordering does. Illustrate it for 2nd term $(-i)^2/2! \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T(H_I(t_1)H_I(t_2)$, get twice the integral over the $t_1 > t_2$ region instead of the integral over the square.

• Now use Wick's theorem to get rid of the time ordered products. Thereby compute probability amplitude for a given process

$$\langle f|(S-1)|i\rangle = \langle f|Te^{-i\int d^4x :\mathcal{H}_I(x):}|i\rangle \equiv i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i).$$

The initial states have momenta $p_1 \dots p_n$ and the final states have momenta $q_1 \dots q_m$. Need to strip off the momentum conserving delta function to get the amplitude. Note the normal ordering in $:\mathcal{H}_I:(x)$: we want the full Hamiltonian to be normal ordered so e.g. $|0\rangle$ has zero energy. This avoids some issues with quantum loops. As you will later see, this normal ordering is accomplished via counterterms.

Dimensional analysis: $[\mathcal{A}_{fi}] = 4 - n_i - n_f$.

• Look at some examples, and connect with Feynman diagrams. As a first, simple example consider the toy model for mesons and (bosonic) baryon, with $H_{int} = \int d^3x g\phi\psi^\dagger\psi$. Note that there is a $\psi \to e^{i\alpha}\psi$ global symmetry, so there is a corresponding conserved current and charge, which we'll call "nucleon number". We choose to assign ψ nucleon number charge -1 and $\psi^\dagger$ nucleon number $+1$.

The $g\phi\psi\bar{\psi}$ interaction is also a toy model for the Higgs' coupling to fermions, where the scalar ϕ's "Yukawa" coupling mediates a force. The strong, weak, and electromagnetic forces are communicated by spin 1 gauge fields. Gravity is mediated by the spin 2 graviton (and the difference between spin 1 vs spin 2 is part of why quantum gravity is conceptually and technically challenging). Spin 0 scalars can also mediate forces, as in this example. We'll see that their force is always attractive (even spins always lead to attractive forces). Fifth force experimental bounds constrain the existence, mass, and couplings of fundamental scalars. In our toy model, where ψ and $\bar{\psi}$ are scalars, the theory has a vacuum instability, since a cubic potential isn't bounded below. This shows up only indirectly in perturbation theory, and is more of a non-perturbative issue. For the actual Yukawa couplings, ψ is instead a Fermion.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E}\left(a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right), \quad \psi(x) = \int \frac{d^3p}{(2\pi)^3 2E}\left(b(p)e^{-ipx} + c^\dagger(p)e^{ipx}\right),$$

We'll say that a annihilate a meson ϕ, b annihilates a nucleon N and $c^\dagger$ creates an anti-nucleon $\bar{N}$. The conserved charge is $Q = N_b - N_c$.

Examples of states:

$$|\phi(p)\rangle = a^\dagger(p)|0\rangle, \qquad |N(p)\rangle = b^\dagger(p)|0\rangle, \qquad |\bar{N}(p)\rangle = c^\dagger(p)|0\rangle.$$

Note then e.g.

$$\langle 0|\phi(x)|\phi(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi(x)|N(p)\rangle = e^{-ip\cdot x}, \qquad \langle 0|\psi^\dagger(x)|N(p)\rangle = 0.$$

Example: meson decay. $|i\rangle = a^\dagger(p)|0\rangle$, $|f\rangle = b^\dagger(q_1)c^\dagger(q_2)|0\rangle$. Compute $\langle f|S|i\rangle = -ig(2\pi)^4\delta^4(p - q_1 - q_2)$ to $\mathcal{O}(g)$, i.e. $\mathcal{A} = -g$. Probability $\sim g^2$.

Comment: draw pictures to illustrate a $\sim g^3$ correction, with 1 loop. In general, amplitudes scale like $(g^2/16\pi^2)^L$ where L is the number of loops. We'll see that loops lead to divergent momenta integrals, eg. $\int^\Lambda d^4k/(k^2 - m^2) \sim \Lambda^2$. This is handled via renormalization (more next quarter).

- Now consider $N + N \to N + N$, to $\mathcal{O}(g^2)$. The initial and final states are

$$|i\rangle = b^\dagger(p_1)b^\dagger(p_2)|0\rangle, \qquad \langle f| = \langle 0|b(p_1')b(p_2').$$

The term that contributes to scattering at $\mathcal{O}(g^2)$ is **(don't forget the time ordering!)**

$$T\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 \phi(x_1)\psi^\dagger(x_1)\psi(x_1)\phi(x_2)\psi^\dagger(x_2)\psi(x_2).$$

The term that contributes to $S - 1$ thus involves

$$\langle p_1'p_2'| : \psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2) : |p_1p_2\rangle = \langle p_1'p_2'| : \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1, p_2\rangle.$$

$$= \left(e^{i(p_1'x_1 + p_2'x_2)} + e^{i(p_1'x_2 + p_2'x_1)}\right)\left(e^{-i(p_1x_1 + p_2x_2)} + e^{-i(p_1x_2 + p_2x_1)}\right).$$

The amplitude involves this times $D_F(x_1 - x_2)$ (from the contraction), with the prefactor and integrals as above. The final result is

$$i(-ig)^2\left[\frac{1}{(p_1 - p_1')^2 - \mu^2 + i\epsilon} + \frac{1}{(p_1 - p_2')^2 - \mu^2 + i\epsilon}\right](2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2').$$