10/9/19 Lecture outline

• Continue from last time. $\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}m^2\phi^2 - \rho\phi$, where ρ is a classical source. Solve by $\phi = \phi_0 + i\int d^4y D(x-y)\rho(y)$, where ϕ_0 is a solution of the homogeneous KG equation and the green's function $D(x-y)$ satisfies

$$(\partial_x^2 + m^2)D(x-y) = -i\delta^4(x-y).$$

By a F.T., get

$$D_?(x-y) = \int_? \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)}.$$

Consider the k_0 integral in the complex plane. The integration contour is not yet defined because it goes through poles at $k_0 = \pm\omega_k$, where $\omega_k \equiv +\sqrt{\vec{k}^2+m^2}$. We must deform the contour to avoid the poles. There are choices about whether the contour goes above or below the poles, and that's what the ? label indicates. Note that $e^{-ik(x-y)} \sim e^{-ik_0(x_0-y_0)}$ will converge for imaginary k_0 if $Im(k_0(x_0-y_0)) < 0$, so for $x_0 > y_0$ we close the contour in the LHP and for $x_0 < y_0$ we close in the UHP. The integrand is analytic modulo the simple poles, so the k_0 integral is easily evaluated by Cauchy's theorem, with contributions coming from whichever of the two poles are inside the closed integration contour.

Going above both poles gives the retarded green's function, $D_R(x-y)$ which vanishes for $x_0 < y_0$. Considering $x_0 > y_0$, get that

$$\begin{aligned} D_R(x-y) &= \theta(x_0-y_0)\int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)} - e^{ik(x-y)}) \\ &\equiv \theta(x_0-y_0)(D_1(x-y) - D_1(y-x)) = \theta(x_0-y_0)\langle[\phi(x),\phi(y)]\rangle, \end{aligned}$$

where $D_1(x-y)$ is as defined in the previous lecture

$$\langle 0|\phi(x)\phi(y)|0\rangle \equiv D_1(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)}e^{-ik(x-y)}.$$

This is what one usually does when solving wave equations with a source: the $\rho(y)$ source only affects $\phi(x)$ in the future. But still there is something weird about it, because the $k_0 = -\sqrt{\vec{k}^2+m^2}$ pole is contributing and that has negative energy. If the k_0 integration contour instead goes below both poles, this gives the advanced propagator, which vanishes for $y_0 < x_0$.

Feynman propagator: go above the $k_0 = E_k$ pole and below the $k_0 = -E_k$ pole. $-E_k$ pole is heuristically the anti-matter, traveling backward in time. Show that this gives the time ordering discussed in the previous lecture:

$$D_F(x-y) \equiv \langle T\phi(x)\phi(y)\rangle = \begin{cases} \langle \phi(x)\phi(y)\rangle & \text{if } x_0 > y_0 \\ \langle \phi(y)\phi(x)\rangle & \text{if } y_0 > x_0 \end{cases}.$$

Here T means to time order: order operators so that earliest is on the right, to latest on left. Object like $\langle T\phi(x_1)\dots\phi(x_n)\rangle$ will play a central role in this class. Time ordering convention can be understood by considering time evolution in $\langle t_f|t_i\rangle$. Evaluate $D_F(x-y)$ by going to momentum space:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik(x-y)},$$

where $\epsilon \to 0^+$ enforces that we go below the $-\omega_k$ pole and above the $+\omega_k$ pole, i.e. we get $D(x-y)$ if $x_0 > y_0$, and $D(y-x)$ if $x_0 < y_0$, as desired from the definition of time ordering. We'll see that this ensures causality.

• The pole placement is such that the contour can be rotated to be along the imaginary k_0 axis, running from $-i\infty$ to $+i\infty$. This will later tie in with a useful way to treat QFT, by going to Euclidean space via imaginary time. It is something of a technical trick, but there is also something deep about it. Analyticity properties of amplitudes are deeply connected with causality.

• Define contraction of two fields $A(x)$ and $B(y)$ by $T(A(x)B(y)) - :A(x)B(y):$. This is a number, not an operator. Let e.g. $\phi(x) = \phi^+(x) + \phi^-(x)$, where ϕ^+ is the term with annihilation operators and ϕ^- is the one with creation operators (using Heisenberg and Pauli's reversed-looking notation). Then for $x^0 > y^0$ the contraction is $[A^+, B^-]$, and for $y^0 > x^0$ it is $[B^+, A^-]$. So can put between vacuum states to get that the contraction is $\langle TA(x)B(y)\rangle$. For example, in the KG theory the contraction of $\phi(x)$ and $\phi(y)$ is $D_F(x-y)$.

• Wick's theorem (we'll soon see it's useful, since S-matrix elements will involve T ordered correlation functions):

$$\begin{aligned} T(\phi_1\dots\phi_n) &= :\phi_1\dots\phi_n: + \sum_{contractions} :\phi_1\dots\phi_n: \\ &= :e^{\frac{1}{2}\sum_{i,j=1}^n C(\phi_i\phi_j)\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}}\phi_1\dots\phi_n: \end{aligned}$$

(where C is the contraction symbol) to get rid of the time ordered products.

Prove Wick's theorem by iteration: define the RHS as $W(\phi_1 \dots \phi_n)$ and we assume $T(\phi_2 \dots \phi_n) = W(\phi_2 \dots \phi_n)$ and want to prove then that $T(\phi_1 \dots \phi_n) = W(\phi_1 \dots \phi_n)$. WLOG, take $t_1 > t_2 \dots t_n$ so $T(\phi_1 \dots \phi_n) = \phi_1 T(\phi_2 \dots \phi_n) = \phi_1 W(\phi_2 \dots \phi_n) = \phi_1^- W + W\phi_1^+ + [\phi_1^+, W]$. The first two terms are normal ordered and give all contractions not involving ϕ_1, while the last gives all normal ordered contractions involving ϕ_1.

So note that

$$\langle T(\phi_1 \dots \phi_n) \rangle \begin{cases} 0 & \text{for } n \text{ odd} \\ \sum_{\text{fullycontracted}} & \text{for } n \text{ even.} \end{cases}$$