10/7/19 Lecture outline

⋆ **Reading: Coleman lectures 2-4. Tong chapters 1-2.**

• Continue from last time. The KG theory has $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$. The EOM are linear (it is a free theory), and are classically solved by plane waves having $p^2 = m^2$. Upon quantization, the states of the theory have a conserved particle number, and the quanta are identical scalar fields of mass m.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3(2\omega(k))}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}].$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3(2\omega)\delta^3(\vec{k} - \vec{k}'),$$

i.e. they are creation and annihilation operators (with our relativistic measure). Create states with momenta p_1^μ, ..., p_n^μ via $a^\dagger(p_1)\dots a^\dagger(p_n)|0\rangle$. Note that these behave as identical bosons: the state is symmetric under exchanging any pair of momenta, because $[a^\dagger(p), a^\dagger(p')] = 0$.

• Write $\phi(x) = \phi^+(x) + \phi^-(x)$, with (backwards looking Heisenberg / Pauli notation)

$$\phi^+(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} a(k)e^{-ikx}, \qquad \phi^-(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega(k)} a(k)^\dagger e^{ikx}$$

where $\phi^\pm$ are positive / negative frequency. Historically, first attempt was to keep just ϕ_+ and regard it as a quantum wavefunction, ψ, with probability $\sim |\psi|^2$. Doesn't work.

The Hamiltonian is

$$H = \int d^3x(\dot{\phi}\Pi - \mathcal{L}) = \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^2(2\omega)}\omega(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})).$$

Normal order the first term. Define $: AB :$ for operators A and B to mean that the terms are arranged so that the annihilation operators are on the right, e.g. $: \phi^+(x)\phi^-(y) := \phi^-(y)\phi^+(x)$. Good for acting on $|0\rangle$.

The vacuum $|0\rangle$ is annihilated by all $a(k)$, and we drop the cosmological constant contribution so $H|0\rangle = 0$ (the zero-point contribution has a $\delta(\vec{k} = 0)$ which can be interpreted as V the volume of spacetime).

Take

$$H \equiv: H := \int \frac{d^3k}{(2\pi)^2(2\omega)}\omega a^\dagger(\vec{k})a(\vec{k}),$$

$$\vec{P} \equiv: \vec{P} := \int d^3x \hat{e}_i T^{i0} = \int \frac{d^3k}{(2\pi)^2(2\omega)}\vec{k}a^\dagger(\vec{k})a(\vec{k}).$$

We're dropping the CC contributing term in H, as discussed last time. So $P^\mu|0\rangle = 0$ and $P^\mu|p_1 \dots p_n\rangle = p^\mu_{tot}|p_1 \dots p_n\rangle$, where $|p_1 \dots p_n\rangle = \prod_n a^\dagger(k_n)|0\rangle$ and $p^\mu_{tot} = \sum_n p^\mu_n$.

• Comment on $\phi(x^\mu)$'s dependence on x^μ. These are operators in the Heisenberg picture, where $\hat{\phi}(x^\mu) = \hat{\mathcal{U}}(x^\mu)\hat{\phi}(0)\hat{\mathcal{U}}(x^\mu)^\dagger$ where we temporarily put hats to emphasize what are operators and $\hat{\mathcal{U}}(x^\mu) = e^{-i\hat{P}_\mu x^\mu}$ is the unitary time and space translation operator.

• We will compute probability amplitudes for scattering processes. E.g. cross sections and decay lifetimes will be of the form (Observable)=$|\langle f|S|i\rangle|^2$ (Phase space factors). The amplitude $\langle f|S|i\rangle$ has initial-state $|i\rangle$ obtained from creation operators acting on the vacuum. The S-matrix elements will be computed from products of operators acting on the vacuum. As a first example, consider the two-point field correlation function:

$$\langle 0|\phi(x)\phi(y)|0\rangle \equiv D_1(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} e^{-ik(x-y)}.$$

Note also that $2i\partial_{x^0} D(x-y)$ is the integral that we saw in last lecture, for the probability amplitude to find a particle having traveled with spacetime displacement $(x-y)^\mu$. For spacelike separation, $(x-y)^2 = -r^2$, we here get $D(x-y) = \frac{m}{2\pi^2 r}K_1(mr)$, with K_1 a Bessel function. Recall that the Bessel function has a simple pole when its argument vanishes, and exponentially decays at infinity. So $D(x-y) \sim \exp(-m|\vec{x}-\vec{y}|)$ is non-vanishing outside the forward light cone.

The above above correlator is not directly a physical observable, and having it not vanish outside the light cone does not immediately imply and causality. There could be observable effects, from interference, if a commutator of fields is non-vanishing outside of the lightcone. Let's show that this does not happen. Note that

$$[\phi(x), \phi(y)] = [\phi^+(x), \phi^-(y)] + [\phi^-(y), \phi^+(x)] =$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \int \frac{d^3k'}{(2\pi)^3 2\omega(k')} [a(k), a^\dagger(k')] e^{-ikx+ik'y} - (x \leftrightarrow y)$$

Note that the commutator is a c-number, not an operator:

$$[\phi(x), \phi(y)] = D_1(x-y) - D_1(y-x),$$

where $D_1(x-y)$ is as defined above. For spacelike separation, $(x-y)^2 = -r^2$, $D_1(x-y) = \frac{m}{2\pi^2 r}K_1(mr)$, with K_1 a Bessel function. For spacelike separation, we can map $(x-y)^\mu$ to $-(x-y)^\mu$ by a Lorentz transformation, so $D_1(x-y) - D_1(y-x) = 0$. Good. The commutator is non-vanishing for timelike separation.

Note that $[\phi(x),\phi(y)]=0$ for $(x-y)^2<0$, wouldn't have been true for just $\phi^+(x)$, so there would be information propagating outside the light cone. Moreover, neither $|\phi|^2$ nor $|\phi^+|^2$ can be interpreted as a conserved probability – the relativistic expression $E=\sqrt{\vec{p}^2+m^2}$ necessarily leads to particle productions. So instead we interpret ϕ as similar to $\vec{x}$ in QM, as a hermitian operator, not a wavefunction.

• Comment (with details to follow): $\langle 0|\phi(x)\phi(y)|0\rangle$ satisfies the equations of motion for either field, except for a contact term when $x^\mu=y^\mu$. Note that $\langle 0| :\phi(x)\phi(y): |0\rangle=0$. The physical observables of QFT actually involve **time ordered** correlation functions of operators $\langle T\phi(x)\phi(y)\rangle\equiv\Theta(x^0-y^0)\langle\phi(x)\phi(y)\rangle+\Theta(y^0-x^0)\langle\phi(y)\phi(x)\rangle$. Often we will drop the T, because we'll just remember that it's always implicitly there.

• Get more interesting theories by adding interactions, e.g. $V(\phi)=\frac{1}{2}m^2\phi^2+\lambda\phi^4$, treat 2nd term as a perturbation. We can consider perturbative solutions in both classical or quantum field theory. The starting point is the green's function for the theory with a forcing function source:

• Consider $\mathcal{L}=\frac{1}{2}\partial\phi^2-\frac{1}{2}m^2\phi^2-\rho\phi$, where ρ is a classical source. Solve the EOM by $\phi=\phi_0+i\int d^4yD(x-y)\rho(y)$, where ϕ_0 is a solution of the homogeneous KG equation and the green's function $D(x-y)$ satisfies

$$(\partial_x^2+m^2)D(x-y)=-i\delta^4(x-y).$$

By a F.T., get

$$D_?(x-y)=\int_?\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)}.$$

Consider the k_0 integral in the complex plane. There are poles at $k_0=\pm\omega_k$, where $\omega_k\equiv+\sqrt{\vec{k}^2+m^2}$. There are choices about whether the k^0 contour goes above or below the poles, and that's what the ? label indicates.

Note that $e^{-ik\cdot(x-y)}=e^{-ik^0(x^0-y^0)+\dots}$ is such that, for $x^0-y^0>0$, we can close the k^0 contour in the LHP, whereas for $x^0-y^0<0$ we close in the UHP.

The retarded green's function, $D_R(x-y)$, by definition vanishes for $x_0<y_0$. We thus get D_R if the k_0 contour goes above both poles: then closing the contour in the UHP gives zero. Going above both poles gives the retarded green's function, $D_R(x-y)$

$$\begin{aligned}D_R(x-y)&=\theta(x_0-y_0)\int\frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik(x-y)}-e^{ik(x-y)})\\&\equiv\theta(x_0-y_0)(D_1(x-y)-D_1(y-x))=\theta(x_0-y_0)\langle[\phi(x),\phi(y)]\rangle,\end{aligned}$$

where $D_1(x-y)$ is as defined above. This is reasonable: the $\rho(y)$ source only affects $\phi(x)$ in the future.

Going below both poles gives the advanced propagator, which vanishes for $y_0 < x_0$.

• Feynman propagator: go above the $k_0 = E_k$ pole and below the $k_0 = -E_k$ pole. $-E_k$ pole is heuristically the anti-matter, traveling backward in time.