10/2/19 Lecture outline

⋆ **Reading: Coleman lectures 2-4. Tong chapters 1-2.**

• Continue from last time. Consider classical, relativistically invariant field theory. The field could e.g. be spin $0, \frac{1}{2}$, or 1 (higher spin is also fine classically, e.g. the metric has spin 2, but the quantum theories have issues and we won't discuss it here). We will start with spin 0 fields, i.e. scalars $\phi_a(t, \vec{x})$, with $S = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a)$. Then $\Pi^\mu_a = \partial\mathcal{L}/\partial(\partial_\mu\phi_a)$, and E.L. eqns $\partial\mathcal{L}/\partial\phi_a = \partial_\mu \Pi^\mu_a$. Define $\Pi_a \equiv \Pi^0_a$. $H = \int d^3x(\Pi\dot{\phi}_a - \mathcal{L}) = \int d^3x \mathcal{H}$. Everything is relativistically invariant if $\mathcal{L}$ is Lorentz invariant.

Example: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$, gives $\Pi = \dot{\phi}$ and $\dot{\Pi} = \nabla^2\phi - m^2\phi$, the Klein-Gordon equation: $(\partial^2 + m^2)\phi = 0$. Can't interpret ϕ as a probability wavefunction because of solutions $E = \pm\sqrt{\vec{p}^2 + m^2}$. But we'll see that the KG equation is fine as a field theory.

The field has both creation and annihilation operators, corresponding to the $E = \pm\sqrt{\vec{k}^2 + m^2}$ solutions. Write general classical solution

$$\phi_{cl}(x) = \int \frac{d^3k}{(2\pi)^3(2\omega(k))}[a_{cl}(k)e^{-ikx} + a^*_{cl}(k)e^{ikx}],$$

where $a_{cl}(k)$ are classical constants of integration, determined by the initial conditions. We'll quantize soon. Another example: $\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \nabla\psi^* \cdot \nabla\psi - m\psi^*\psi$. Get EOM: $i\partial_t\psi = -\nabla^2\phi + m\psi$. Looks like S.E., but again don't want to interpret ψ as a probability amplitude – here it's a field, that we can consider quantizing. This example won't work for ψ a scalar field, but we'll later consider an analogous theory where ψ is a fermion field, and the equation is the Dirac equation.

• The normalization of the momentum space integral is chosen to be relativistically nice: it's Lorentz invariant: $d^3k/\omega = d^3k'/\omega'$. Here's why: $d^4k\delta(k^2 - m^2)\theta(k_0) \to \frac{d^3k}{2\omega(k)}$ upon doing the k_0 integral. So normalize $\langle k'|k\rangle = (2\pi)^3 2\omega(k)\delta^3(\vec{k} - \vec{k}')$, with $|k\rangle \equiv \sqrt{(2\pi)^3 2\omega_k}|\vec{k}\rangle$.

• In field theory, as in particle mechanics, continuous symmetries lead to conservation laws, via Noether's theorem. If a variation $\delta\phi_a$ changes $\delta\mathcal{L} = \partial_\mu F^\mu$, then it's a symmetry of the action and there is a conserved current: $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu$.

Example: $x^\mu \to x^\mu + \epsilon^\mu$, $\delta\phi_a = \epsilon^\nu\partial_\nu\phi_a$, $\delta\mathcal{L} = \epsilon^\nu\partial_\nu\mathcal{L}$ (assuming no explicit x dependence). Get $T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\nu\phi_a - g_{\mu\nu}\mathcal{L}$. Stress energy tensor. Conserved charge is $P_\mu = \int d^3\vec{x}T_{\mu 0}$.

Another example: $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, leads to conserved $M_{\mu\rho\sigma} = x_\mu T_{\rho\sigma} - x_\sigma T_{\rho\mu}$. Conserved charge is $M_{\rho\sigma} = \int d^3x M_{0\rho\sigma}$. Conserved angular momentum.

• Canonical quantization: generalize QM by replacing $q_a(t) \to \phi_a(t, \vec{x})$. QM is like QFT in zero spatial dimensions, with the field playing role of position before:

$$[\phi_a(\vec{x},t), \Pi_b(\vec{y},t)] = i\delta_{ab}\delta^3(\vec{x}-\vec{y}) \quad \textit{(Equal time commutators).}$$

• Consider the KG equation in $0+1$ dimensions, i.e. the SHO: $L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2 x^2$, $p = \partial L/\partial\dot{\phi} = \dot{\phi}$. Classical EOM solved by $x_{cl} = ae^{-i\omega t} + a^* e^{i\omega t}$. Now quantize: $[x, p] = i\hbar$, $[a, a^\dagger] = 1$, $H = \omega(a^\dagger a + \frac{1}{2})$. In the Heisenberg picture, $\widehat{x} = \sqrt{\frac{1}{2\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t})$; $p = \dot{x} = i\sqrt{\frac{\omega}{2}}(ae^{i\omega t} - a^\dagger e^{-i\omega t})$.

• Now quantize the KG field theory in $3+1$ dimensions. Write

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3(2\omega(k))}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}].$$

Then canonical quantization implies that

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3(2\omega)\delta^3(\vec{k}-\vec{k}'),$$

i.e. they are creation and annihilation operators (with our relativistic measure). The Hamiltonian is then

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^2(2\omega)}\omega(a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})).$$

Need to normal order the first term. Define $: AB :$ for operators A and B to mean that the terms are arranged so that the annihilation operators are on the right, so annihilates the vacuum.

• The vacuum $|0\rangle$ is annihilated by all $a(k)$. Create states with momenta p_1^μ, …, p_n^μ via $a^\dagger(p_1)\dots a^\dagger(p_n)|0\rangle$. Note that these behave as identical bosons: the state is symmetric under exchanging any pair of momenta, because $[a^\dagger(p), a^\dagger(p')] = 0$.