11/27/19 Lecture outline

- Continue from last time. We quantized the Fermion field

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b^r(p)u^r(p)e^{-ipx} + c^{r\dagger}(p)v^r(p)e^{ipx}\right)$$

where $(\not{p} - m)u^s(p) = 0$, $(\not{p} + m)v^r(p) = 0$, and

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m$$

first by canonical quantization with equal time anticommutation relations, and then

$$\{b^r(p), b^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec{p} - \vec{p}'), \qquad \{c^r(p), c^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec{p} - \vec{p}').$$

Alternatively, we can use the path integral. Let $\psi(x)$ and $\bar{\psi}(x)$ be Grassmann valued (anticommuting) functions (vs operators in canonical quantization). The path integral partition function for sources $\alpha(x)$ and $\bar{\alpha}(x)$ is

$$Z[\alpha(x), \bar{\alpha}(x)] = \mathcal{N} \int [d\psi(x)][d\bar{\psi}(x)] \exp\left(\frac{i}{\hbar}\int d^4x[\mathcal{L} + \bar{\alpha}(x)\psi(x) + \bar{\psi}(x)\alpha(x)]\right).$$

where $\mathcal{N}^{-1}$ is the vacuum bubble normalization such that $Z[0,0] = 1$. The Grassmann version of the Gaussian integral is

$$\int d\Theta d\bar{\Theta} \exp[i(\bar{\Theta}, A\Theta) + i(\bar{\alpha}, \Theta) + i(\bar{\Theta}, \alpha)] = \det(iA)\exp(-i(\bar{\alpha}, A^{-1}\alpha).$$

Thus for the case of the free Dirac equation we get ($\psi/\hbar \to \Theta$ and $(i\not{\partial} - m)\hbar \to A$)

$$Z_{Dirac}[\alpha, \bar{\alpha}] = \exp\left(-\frac{1}{\hbar}\int d^4x d^4y \bar{\alpha}(x)S(x-y)\alpha(y)\right)$$

where

$$(i\not{\partial}_x - m)S(x-y) = i\delta^4(x-y) \qquad \text{so} \qquad S(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{\not{p} - m + i\epsilon}e^{-ip(x-y)}.$$

Then see e.g.

$$\langle T\psi(x)\bar{\psi}(y)\rangle = (\frac{\hbar}{i})^2 \frac{\delta}{\delta\bar{\alpha}(x)}\frac{\delta}{\delta\alpha(y)} = \hbar S(x-y).$$

• Consider perturbation theory for an interacting theory. As an example, consider our old friend toy model of nucleons and pions, now upgraded to be more realistic by making the nucleons Fermions. Applications: this is Yukawa's original model for explaining the attraction between nucleons. It works. We'll see how the potential is always attractive, whether the nucleon charges are the same or opposite sign. This model will also set the stage for quantum electrodynamics (QED), where the scalar meson is replaced with the spin 1 photon and the nucleons are replaced with electrons. Here the rule that opposites attract and same sign charges repel comes from the difference between spin 1 vs spin 0 force carries. Finally, this model illustrates how the Higgs scalar interacts with the fundamental fermions of Nature.

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 - g\phi\bar{\psi}_a\Gamma_{ab}\psi b.$$

• Let's call the particle states nucleons and anti-nucleons (we could also call them electrons and positrons etc):

$$|N(p,r)\rangle = b(p)^{r\dagger}|0\rangle \qquad |\bar{N}(p,r)\rangle = c^{r\dagger}(p)|0\rangle.$$

Then

$$\langle 0|\psi(x)|N(p,r)\rangle = e^{-ipx}u^r(p), \qquad \langle N(p,r)|\bar{\psi}(x)|0\rangle = e^{ipx}\bar{u}^r(p).$$

Incoming fermions get a factor of $u^r(p)$, outgoing fermions get $\bar{u}^r(p)$; incoming antifermions gets $\bar{v}^r(p)$, and outgoing antifermions get $v^r(p)$. Write the amplitude by following the arrows backwards, from the head to the tail. We can think of the amplitudes either in terms of our original description using Dyson's formula and Wick contractions, or in terms of the LSZ description.

• Compute amplitudes in the (spin $\frac{1}{2}$) nucleon + (scalar) meson toy model.

Tinkertoy pieces:

$$\mathcal{L} \supset \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad \to \qquad \text{fermion propagator:} \qquad \frac{i}{p\!\!\!/ - m + i\epsilon},$$

$$\mathcal{L} \supset \tfrac{1}{2}\partial\phi\partial\phi - \tfrac{1}{2}\mu^2\phi^2 \qquad \to \qquad \text{scalar propagator:} \qquad \frac{i}{p^2 - \mu^2 + i\epsilon},$$

$$\mathcal{L} \supset -g\phi\bar{\psi}_a\Gamma_{ab}\psi_b(x) \qquad \to \qquad \text{scalar, fermion vertex} \qquad -ig\Gamma,$$

where the index a, b runs over the four fermion components (spin up and down for fermion and anti-fermion), so Γ is a 4×4 matrix (natural choices are $\Gamma = 1_{4\times 4}$ or $\Gamma = i\gamma_5$, where recall $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, and the i is there to keep $\mathcal{L}^\dagger = \mathcal{L}$, since $(\gamma^0\gamma_5)$ is anti-hermitian).

Incoming fermions get a factor of $u^r(p)$, outgoing fermions get $\bar{u}^r(p)$; incoming antifermions gets $\bar{v}^r(p)$, and outgoing antifermions get $v^r(p)$. The amplitude has indices $r = 1, 2$ for each external fermion, which accounts for the external fermion's spin. For internal fermion propagators we sum over the four fermion indices, which is accomplished by matrix multiplication of the above tinkertoy pieces, with Tr put in as appropriate. Write the amplitude by following the arrows backwards, from the head to the tail. The propagator for an anti-Fermion has the arrow reversed and by CPT the propagator is related to the Fermion's propagator by $\not{p} \to -\not{p}$.

• Minus sign of fermion loop. This follows from working through the Dyson/Wick procedure, accounting for the minus signs when fermions are exchanged, as needed to bring contracted fermions next to each other. This relative minus sign for fermion vs boson loops plays a big role in supersymmetry.

• Examples of amplitudes, computed to lowest non-trivial order: $\phi(p_1) \to N^{r_1}(p_1') + \bar{N}^{r_2}(p_2')$, get $i\mathcal{A} = \bar{u}^{r_1}(p_1')(-ig\Gamma)v^{r_2}(p_2')$. Write out also e.g. $\phi(p_2) + N^{r_1}(p_1) \to N^{r_2}(p_1')$ gives $i\mathcal{A} = \bar{u}^{r_2}(p_1')(-ig\Gamma)u^{r_1}(p_1)$ vs $\phi(p_2) + \bar{N}^{r_1}(p_1) \to \bar{N}^{r_2}(p_1')$ has $u \to v$ and a relative minus sign. Now consider $N^r(p) + \phi(q) \to N^{r'}(p') + \phi(q')$:

$$i\mathcal{A} = (-ig)^2 \bar{u}^{r'}(p')\Gamma \left(\frac{i(\not{p} + \not{q} + m)}{(p+q)^2 - m^2 + i\epsilon} + \frac{i(\not{p} - \not{q}' + m)}{(p-q')^2 - m^2 + i\epsilon} \right) \Gamma u^r(p).$$

$\bar{N} + \phi \to \bar{N} + \phi$:

$$i\mathcal{A} = -(-ig)^2 \bar{v}^r(p)\Gamma \left(\frac{i(-\not{p} - \not{q} + m)}{(p+q)^2 - m^2 + i\epsilon} + \frac{i(-\not{p} + \not{q}' + m)}{(p-q')^2 - m^2 + i\epsilon} \right) \Gamma v^{r'}(p').$$

$N + N \to N + N$:

$$i\mathcal{A} = -ig^2 \left(\frac{\bar{u}^{s'}_{q'}\Gamma u^s_q \bar{u}^{r'}_{p'}\Gamma u^r_p}{(q-q')^2 - \mu^2 + i\epsilon} - \frac{\bar{u}^{s'}_{q'}\Gamma u^r_p \bar{u}^{r'}_{p'}\Gamma u^s_q}{(q-p')^2 - \mu^2 + i\epsilon} \right).$$

• Attractive Yukawa potential for both $\psi\psi \to \psi\psi$, and also $\psi\bar{\psi} \to \psi\bar{\psi}$. Recall $\mathcal{A}_{NR} = -i\int d^3\vec{r} e^{-i(\vec{p}'-\vec{p})\cdot\vec{r}} U(\vec{r})$. For $\psi\psi \to \psi\psi$, $\mathcal{A}_{NR} \supset -i(-ig)^2(2m)\frac{1}{(\vec{p}-\vec{p}')^2+\mu^2}$ when the spins are unchanged. Gives $U(\vec{r}) = -g^2 e^{-\mu r}/4\pi r$. For $\psi\bar{\psi} \to \psi\bar{\psi}$, amplitude differs by sign, but so does $\bar{v}v$, so again get attractive potential.

• Summing over polarizations for inclusive rates, and simplifications using

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m.$$

• Example $\Gamma = i\gamma_5$, $N + \phi \to N + \phi$, simplify $i\mathcal{A}$. Compute $|\mathcal{A}|^2$ and average over initial spins and sum over final spins. Simplify.

$$i\mathcal{A} = ig^2 \bar{u}_{p'}^{r'}\gamma_5 \left(\frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2 + i\epsilon} + \frac{\not{p} - \not{q}' + m}{(p-q')^2 - m^2 + i\epsilon} \right) \gamma_5 u_p^r,$$

$$i\mathcal{A} = ig^2 \bar{u}^{(r')}(p')\not{q}u^{(r)}(p)F, \qquad F \equiv \left[\frac{1}{2p\cdot q + \mu^2 + i\epsilon} + \frac{1}{2p'\cdot q + \mu^2 + i\epsilon} \right].$$

$$|\mathcal{A}|^2 = g^4 F^2 q_\mu q_\nu Tr[\bar{u}(p')^{r'}\gamma^\mu u(p)^r \bar{u}(p)^r \gamma^\nu u(p)^{r'}].$$

Average over initial spins and sum over final ones (often physically relevant, and it simplifies the expression, using the completeness relations)

$$\frac{1}{2}\sum_{r,r'} |\mathcal{A}|^2 = \frac{1}{2} g^2 F^2 q_\mu q_\nu Tr[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu]$$
$$= 2g^4 F^2 [2(p'\cdot q)(p\cdot q) - p\cdot p'\mu^2 + m^2\mu^2].$$