11/25/19 Lecture outline

- Recall from last time, the Dirac Lagrangian, whose EOM is the Dirac equation

$$S_{Dirac} = \int d^4x \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) \qquad \rightarrow \qquad \text{EOM} \quad (i\partial\!\!\!/ - m)\psi = 0.$$

(Review the slash notation and e.g. $\partial\!\!\!/^2 = \partial^2 \mathbf{1}$). Dirac wrote this down by thinking about how to make sense of the square-root of the operator appearing in the KG equation, $\sqrt{\partial_\mu \partial^\mu + m^2}$; indeed, $-(i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m) = \partial^2 + m^2$.

The plane wave solutions of the Dirac equation are

$$\psi = u^s(p)e^{-ipx}, \qquad \psi = v^r(p)e^{ipx},$$

where

$$(p\!\!\!/ - m)u^s(p) = 0, \qquad (p\!\!\!/ + m)v^r(p) = 0.$$

If we wanted to solve the eigenvalue equation $p\!\!\!/ X = \lambda X$, we'd find four eigenvalues, and four linearly independent eigenvectors, which form a basis. Here, because $p\!\!\!/^2 = m^2$, we see that $\lambda = \pm m$, so there are two eigenvectors with $\lambda = m$, i.e. u^s, and two with $\lambda = -m$, i.e. v^r. Here r, s both run over $1, 2$, labeling the four eigenvectors, each of which is a 4-component vector. These form a complete, orthogonal basis, with

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m.$$

We'll see how to evaluate Feynman diagrams involving fermions using just these relations. These relations are basis - independent. Explicit expressions for u^r and v^s are less useful and are also basis dependent.

For example, in the Dirac basis:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

in the rest frame of a massive fermion, we get

$$u^{(1)} = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{2m} \\ 0 \\ 0 \end{pmatrix}$$

which can be boosted to get the solution for general p^μ. For the massless case,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar\sigma}\xi^s \end{pmatrix}, \qquad v^r(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^r \\ -\sqrt{p\cdot\bar\sigma}\eta^r \end{pmatrix},$$

where $\xi^\dagger\xi = \eta^\dagger\eta = 1$, and r, s label the basis choices, e.g $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

• The general solution of the classical EOM is a superposition of these plane waves:

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b^r(p) u^r(p) e^{-ipx} + c^{r\dagger}(p) v^r(p) e^{ipx} \right)$$

The theory is quantized by using $\Pi^0_\psi = \partial\mathcal{L}/\partial(\partial_0\psi) = i\psi^\dagger$ and imposing

$$\{\psi(t,\vec x), \Pi(t,\vec y)\} = i\delta(\vec x - \vec y), \qquad \text{i.e.} \qquad \{\psi(t,\vec x), \psi^\dagger(t,\vec y)\} = \delta^3(\vec x - \vec y).$$

Alternatively, we can quantize via a path integral; see below.

If we were to quantize with a commutator rather than anticommutator, get a Hamiltonian that is unbounded below, with c creating antiparticles with negative energy. Shows that spin $\frac{1}{2}$ must have fermionic statistics, to avoid unitarity problems. This is a special case of the general spin-statistics theorem: unitarity requires integer spin fields to be quantized as bosons (commutators) and half-integer spin to be quantized according to Fermi-Dirac statistics (anti-commutators). Leads to the Pauli exclusion principle.

So the coefficients in the plane wave expansion get quantized to be annihilation and creation operators as

$$\{b^r(p), b^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec p - \vec p'), \qquad \{c^r(p), c^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec p - \vec p'),$$

with all other anticommutators vanishing.

• Aside on dimensional analysis $[\psi] = 3/2$, $[u] = [v] = 1/2$, $[b] = [c] = -1$.

• Hamiltonian of the Dirac equation, with fermionic statistics, $\mathcal{H} = \Pi_\psi\dot\psi - \mathcal{L} = \bar\psi(-i\partial_j\gamma^j + m)\psi$, and then $H = \int d^3x \mathcal{H}$ gives

$$: H := \int \frac{d^3p}{(2\pi)^3 2E_p} E_p (b^{r\dagger}(p) b^r(p) + c^{r\dagger}(p) c^r(p)),$$

good, $b^{r\dagger}(p)$ creates a spin 1/2 particle of positive energy, and $c^{r^\dagger}(p)$ creates a spin 1/2 particle of positive energy. The second term was re-ordered according to normal ordering – the terms originally work out to have the opposite order and the opposite sign. Fermionic statistics gives the sign above, upon normal ordering, but Bose statistics would have given the $c^{r\dagger}c^r$ term with a minus sign, leading to H that is unbounded below. We need Fermionic statistics for spin 1/2 fields to get a healthy theory.

• Do perturbation theory as before, but account for Fermi statistics, e.g. $T(\psi(x_1)\psi(x_2)) = -T(\psi(x_2)\psi(x_1))$ and likewise for normal ordered products. Anytime Fermionic variables are exchanged, pick up a minus sign (and sometimes the additional term if the anti-commutator is non-zero). Consider in particular the propagator

$$\{\psi(x), \bar{\psi}(y)\} = (i\partial\!\!\!/_x + m)(D(x-y) - D(y-x)).$$

The Green's function for $(i\partial\!\!\!/_x - m)$ is the $\psi(x)\bar{\psi}(y)$ contraction (time ordered minus normal ordered as before, and it is proportional to the unit operator so we can take expectation value and then the normal ordered part vanishes)

$$\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)}.$$

Vanishes for spacelike separated points. The momentum space fermion propagator is

$$\frac{i}{p\!\!\!/ - m + i\epsilon}.$$

The contraction of $\psi(x)\bar{\psi}(y)$ is $T(\psi(x)\bar{\psi}(y)) -\, :\psi(x)\bar{\psi}(y):$ and can be shown to be a c-number (analogous to the scalar field case). So it is the same as its vacuum expectation value, and thus is the same as $\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle$.

• Generating functional and path integral: Let $\psi(x)$ and $\bar{\psi}(x)$ be Grassmann valued (anticommuting) functions (vs operators in canonical quantization). The path integral partition function for sources $\alpha(x)$ and $\bar{\alpha}(x)$ is

$$Z[\alpha(x), \bar{\alpha}(x)] = \mathcal{N}\int [d\psi(x)][d\bar{\psi}(x)]\exp\left(\frac{i}{\hbar}\int d^4x[\mathcal{L} + \bar{\alpha}(x)\psi(x) + \bar{\psi}(x)\alpha(x)]\right).$$

where $\mathcal{N}^{-1}$ is the vacuum bubble normalization such that $Z[0,0] = 1$. The Grassmann version of the Gaussian integral is

$$\int d\Theta d\bar{\Theta}\exp[i(\bar{\Theta}, A\Theta) + i(\bar{\alpha}, \Theta) + i(\bar{\Theta}, \alpha)] = \det(iA)\exp(-i(\bar{\alpha}, A^{-1}\alpha).$$

Thus for the case of the free Dirac equation we get ($\psi/\hbar \to \Theta$ and $(i\partial\!\!\!/ - m)\hbar \to A$)

$$Z_{Dirac}[\alpha, \bar{\alpha}] = \exp\left(-\frac{1}{\hbar}\int d^4x d^4y \bar{\alpha}(x)S(x-y)\alpha(y)\right)$$

where

$$(i\partial\!\!\!/_x - m)S(x-y) = i\delta^4(x-y) \qquad \text{so} \qquad S(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p\!\!\!/ - m + i\epsilon}e^{-ip(x-y)}.$$

Then see e.g.

$$\langle T\psi(x)\psi(y)\rangle = (\frac{\hbar}{i})^2\frac{\delta}{\delta\bar{\alpha}(x)}\frac{\delta}{\delta\alpha(y)} = \hbar S(x-y).$$