11/20/19 Lecture outline

• Next topic: non-scalar fields (e.g. Fermions or spin 1 gauge fields).

Under lorentz transformations $x^\mu \to x^{\mu'} = \Lambda^\mu_\nu x^\nu$, scalar fields transform as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. Vector fields transform as $A^\mu \to \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$. Generally, $\phi^a \to D[\Lambda]^a_b \phi^b(\Lambda^{-1}x)$, where $D[\Lambda]$ is a rep of the Lorentz group, $D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2]$.

Recall that rotations in 3d can be written as $U(\vec{\phi}) = e^{i\vec{J}\cdot\vec{\phi}/\hbar}$ and when acting on a spin $j$ state $\vec{J} \to \hbar\vec{T}_j$ where $T_j$ are the $(2j+1)$ dimensional representations of the rotation group. For integer $j$, the rotation group is called $SO(3)$; more generally, for any $j$, it is $SU(2)$. The $j = \frac{1}{2}$ representation is the fundamental rep of $SU(2)$ (all others can be obtained by tensor products) and fundamental Fermions, e.g. the electron, are in that rep.

The Lorentz group is similar. In fact, if we go to Euclidean space (e.g. replacing $t \to it$) it is $\cong SO(4) \cong SU(2)_L \times SU(2)_R$. The representations are labeled by $(j_L, j_R)$. The representation $(\frac{1}{2}, \frac{1}{2})$ is a Lorentz vector, e.g. $A^\mu$. The basic spinor representation is $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$. For massless Fermions, the smallest Lorentz representation can be either $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$; these are called left or right handed chiral, two-component Fermions. For massive Fermions one needs both, and that is called a four-component Dirac Fermion.

• [Intend to skip most of this in the lecture.]

Write $D[\Lambda] = \exp(i\frac{1}{2}\Omega_{\mu\nu}\mathcal{M}^{\mu\nu})$, which is a rep if $\mathcal{M}^{\nu\nu}$ satisfies the Lie algebra commutation relation $[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\mu\nu}] = i\eta^{\sigma\mu}\mathcal{M}^{\rho\nu} \pm 3perms$, where the perms account for $\mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}$. E.g. the fundamental rep has $i(\mathcal{M}^{\mu\nu})^{\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}$.

Write the Lorentz transformation generators in terms rotation, whose generators are the angular momentum $\vec{J}$, where $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$, and boosts, with $\vec{K}$ and $K_i = M^{i,0}$. They are similar, e.g. boosting along the $x$ axis vs rotation around the $x$ axis:

$$\Lambda_{boost} = \begin{pmatrix} \cosh\phi & \sinh\phi & & \\ \sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \Lambda_{rotate} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{pmatrix}.$$

So define $\vec{N}^\pm \equiv \frac{1}{2}(\vec{J} \pm i\vec{K})$. Then the Lorentz algebra becomes simply $[N_i^\pm, N_k^\pm] = i\epsilon_{ijk}N_k^\pm$, and $[N^\pm, N_j^\mp] = 0$, i.e. two copies of the familiar rotation commutation relations. The reps are then labeled by $(n_L, n_R)$, where $n_L$ and $n_R$ are non-negative half-integers, like the angular momentum $j$. Note that parity exchanges $\vec{N} \leftrightarrow \vec{N}^\dagger$, so it exchanges the above left and right, hence their names. The angular momentum $\vec{J} = \vec{N} + \vec{N}^\dagger$, so $j$ runs from

$|n_L - n_R|$ to $n_L + n_R$ The scalar rep is $(0,0)$, the vector rep is $(1/2,1/2)$ [1] The basic spinor reps are $(1/2,0)$ and $(0,1/2)$, denoted $u_\pm$; these are called left and right handed Weyl spinors. They both have $D = e^{-i\vec{\sigma}\cdot\hat{e}\theta/2}$ for a rotation by $\theta$ around the $\hat{e}$ axis, but they have $D_\pm = e^{\pm\vec{\sigma}\cdot\hat{e}\phi/2}$ for a boost along the $\hat{e}$ axis, where $v = \tanh\phi$. These 2-component Weyl spinor representations individually play an important role in non-parity invariant theories, like the weak interactions. Parity ($(t,\vec{x}) \to (t,-\vec{x})$) exchances them. So, in parity invariant theories, like QED, they are combined into a 4-component Dirac spinor, $(1/2,0)+(0,1/2)$:

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}.$$

The 4-component spinor rep starts with the clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, e.g.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

There are other choices of reps of the clifford algebra.

$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\eta^{\mu\nu}$, satisfies the Lorentz Lie algebra relation. Under a rotation, $S^{ij} = -\frac{i}{2}\epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$, so taking $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$ get under rotations

$$S[\vec{\varphi}] = \begin{pmatrix} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix}.$$

Under boosts, $\Omega_{i,0} = \phi_i$,

$$S[\Lambda] = \begin{pmatrix} e^{\vec{\phi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\phi}\cdot\vec{\sigma}/2} \end{pmatrix}.$$

This exhibits the 2-component reps that we described above.

- Under Lorentz transformations, spinors transform as $\psi(x) \to S[\Lambda]\psi(\Lambda^{-1}x)$, and $\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger$. Note that $S[\Lambda]^\dagger S[\Lambda] \neq 1$, but $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma_0$. So define $\bar{\psi}(x) \equiv \psi^\dagger\gamma^0$ and note that $\bar{\psi}\psi$ transforms as a scalar, and $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz 4-vector.

For 2-component spinors, $u_-^\dagger\sigma^\mu u_-$ and $u_+^\dagger\bar{\sigma}^\mu u_+$ transform like vectors, where $\sigma^\mu = (1,\sigma^i)$ and $\bar{\sigma}^\mu = (1,-\sigma^i)$. Here are two Lorentz scalars (exchanged under parity): $u_\pm^\dagger u_\mp$.

[1] Consider $\sigma^\mu = (1,\sigma^i)$, where each entry is a $2\times 2$ matrix. Now form $X = x^\mu\sigma^m u$. Lorentz transformations act as $X \to X' = DXD^\dagger$, where $D \in SL(2,C)$. Here $D = e^{-i\vec{\sigma}\cdot\hat{e}\theta/2}$ for a rotation by $\theta$ around the $\hat{e}$ axis, and $D_\pm = e^{\pm\vec{\sigma}\cdot\hat{e}\phi/2}$ for a boost along the $\hat{e}$ axis, where $v = \tanh\phi$. This illustrates the statement that the vector representation of the Lorentz group is $D^{(1/2,1/2)}$.

$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$, anticommutes with all other $\gamma^\mu$ and $(\gamma^5)^2 = 1$. In our above representation of the gamma matrices, $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ are projection operators, projecting on to $u_\pm$.

- The Dirac action:

$$S = \int d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$
$$= \int d^4x (u_+^\dagger i\sigma^\mu \partial_\mu u_+ + u_-^\dagger i\bar{\sigma}^\mu \partial_\mu u_- - m(u_+^\dagger u_- + u_-^\dagger u^+)).$$

The last line exhibits something interesting: if there is a mass term, it is necessary to have both $u_R$ and $u_L$ (preserving parity, which takes $\vec{x} \to -\vec{x}$ and exchanges $u_R \leftrightarrow u_L$). If $m = 0$, we can consider $P$ non-invariant theories with only $u_R$ or only $u_L$. More about this perhaps in a later quarter. Also, the action has a global $U(1)$ symmetry under $\psi \to e^{i\alpha}\psi$, whose Noether conserved charge is fermion number. If $m = 0$, this symmetry is enhanced to $U(1)_R \times U(1)_L$, acting separately on $u_R$ and $u_L$. Neat point: this enhanced symmetry helps explains why the known fermion masses are small. Call $U(1)_V \cong U(1)_R + U(1)_L$ and $U(1)_A \cong U(1)_R - U(1)_L$. Also, call $u_R \equiv u_+$ and $u_L \equiv u_-$.

Starting from the above Lagrangian, we get the EL equations from minimizing the action. Vary $\mathcal{L}$ w.r.t. $\bar{\psi}$ to get the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

Dirac wrote this down by thinking about how to make sense of the square-root of the operator appearing in the KG equation, $\sqrt{\partial_\mu \partial^\mu + m^2}$; indeed, $-(i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m) = \partial^2 + m^2$.

The conjugate momentum to $\psi$ is

$$\pi_\psi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu.$$

So $\psi$ has 4 (rather than 8) real d.o.f., it is the phase-space that has 8 d.o.f.

Let's first consider the plane wave solutions for a single Wey spinor $u_+$, in the $m = 0$ case, so the EOM is $\partial_\mu \sigma^\mu u_+(x) = 0$. Take positive energy, $k_0 = +\sqrt{\vec{k}^2}$, and then the plane wave solutions are

$$u_+(x) = u_+ e^{-ikx}, \qquad \text{or} \qquad u_+(x) = v_+ e^{ikx}.$$

When we quantize, $u_+$ will go with a particle annihilation operator, and $v_+$ will go with an antiparticle creation operator. Plugging into the EOM, $(k_0 - \vec{\sigma} \cdot \vec{k})u_+ = 0$. Taking $\vec{k} = k_0 \hat{z}$, get

$$\langle 0 | u_+(x) | k \rangle \propto e^{-ikx} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note also that the state $|k\rangle$ has spin $J_z = 1/2$, under a rotation by $\theta$ around the $\widehat{z}$ axis, it picks up a phase $e^{i\theta/2}$. The state $|k\rangle$ thus carries helicity $+1/2$, and the annihilation operator that goes with $u_+$ annihilates that state. Likewise

$$\langle k | v_+(x) | 0 \rangle \propto e^{ikx} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so $v_+$ goes with a creation operator creating states of angular momentum $-1/2$ along the direction of motion, i.e. helicity $-\frac{1}{2}$. The theory has particles of helicity $1/2$ and antiparticles of helicity $-1/2$. This can only happen for massless fermions, since otherwise could get the opposite helicity in a boosted frame. Neutrinos are like this. All neutrinos are "left-handed". Nice story from Feynman Lectures on Physics about shaking hands with an alien. (At the time the story was written, it was known that C and P are separately broken, but thought that CP was a valid symmetry. CP would be a valid symmetry if there were only two matter generations in the Standard Model. Now we know that there are actually three generations, and that CP is also violated, by tiny effects. Lorentz symmetry implies that CPT is a valid symmetry, so CP violation is equivalent to breaking of time reversal symmetry at the microscopic level.)

The plane wave solutions of the Dirac equation are

$$\psi = u^s(p)e^{-ipx}, \qquad \psi = v^r(p)e^{ipx},$$

where

$$(\gamma^\mu p_\mu - m)u^s(p) = 0, \qquad (\gamma_\mu p^\mu + m)v^r(p) = 0.$$

If we wanted to solve the eigenvalue equation $\gamma_\mu p^\mu X = \lambda X$, we'd find four eigenvalues, and four linearly independent eigenvectors, which form a basis. Here, because $\not{p}^2 = m^2$, we see that $\lambda = \pm m$, so there are two eigenvectors with $\lambda = m$, i.e. $u^s$, and two with $\lambda = -m$, i.e. $v^r$. Here $r, s$ both run over $1, 2$, labeling the four eigenvectors, each of which is a 4-component vector.

The important properties are that these form a complete, orthogonal basis, with

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs}, \qquad \bar{u}^r v^s = \bar{v}^r u^s = 0.$$

$$\sum_{r=1}^{2} u^r(p)\bar{u}^r(p) = \gamma^\mu p_\mu + m, \qquad \sum_{r=1}^{2} v^r(p)\bar{v}^r(p) = \gamma^\mu p_\mu - m.$$

We'll see how to evaluate Feynman diagrams involving fermions using just these relations. These relations are basis - independent. Explicit expressions for $u^r$ and $v^s$ are less useful and are also basis dependent.

For example, in the Dirac basis:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

in the rest frame of a massive fermion, we get

$$u^{(1)} = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{2m} \\ 0 \\ 0 \end{pmatrix}$$

which can be boosted to get the solution for general $p^\mu$. For the massless case,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \qquad v^r(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^r \\ -\sqrt{p\cdot\bar{\sigma}}\eta^r \end{pmatrix},$$

where $\xi^\dagger\xi = \eta^\dagger\eta = 1$, and $r, s$ label the basis choices, e.g $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- The general solution of the classical EOM is a superposition of these plane waves:

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \left( b^r(p)u^r(p)e^{-ipx} + c^{r\dagger}(p)v^r(p)e^{ipx} \right)$$

The theory is quantized by using $\Pi_\psi^0 = \partial\mathcal{L}/\partial(\partial_0\psi) = i\psi^\dagger$ and imposing

$$\{\psi(t,\vec{x}), \Pi(t,\vec{y})\} = i\delta(\vec{x}-\vec{y}), \qquad \text{i.e.} \qquad \{\psi(t,\vec{x}), \psi^\dagger(t,\vec{y})\} = \delta^3(\vec{x}-\vec{y}).$$

Alternatively, we can quantize via a path integral $Z[\eta,\bar{\eta}] = \int [d\psi][d\bar{p}si] \exp(\frac{i}{\hbar}(S + \int d^4x + \bar{\eta}\psi + \bar{\psi}\eta)$.

If we were to quantize with a commutator rather than anticommutator, get a Hamiltonian that is unbounded below, with $c$ creating antiparticles with negative energy. Shows that spin $\frac{1}{2}$ must have fermionic statistics, to avoid unitarity problems. This is a special case of the general spin-statistics theorem: unitarity requires integer spin fields to be quantized as bosons (commutators) and half-integer spin to be quantized according to Fermi-Dirac statistics (anti-commutators). Leads to the Pauli exclusion principle.

So the coefficients in the plane wave expansion get quantized to be annihilation and creation operators as

$$\{b^r(p), b^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec{p} - \vec{p}'), \qquad \{c^r(p), c^{s\dagger}(p')\} = \delta^{rs}(2\pi)^3 2E_p \delta^3(\vec{p} - \vec{p}'),$$

with all other anticommutators vanishing.

• Aside on dimensional analysis $[\psi] = 3/2$, $[u] = [v] = 1/2$, $[b] = [c] = -1$.

• Hamiltonian of the Dirac equation, with fermionic statistics, $\mathcal{H} = \Pi_\psi \dot{\psi} - \mathcal{L} = \bar{\psi}(-i\partial_j \gamma^j + m)\psi$, and then $H = \int d^3x \mathcal{H}$ gives

$$: H := \int \frac{d^3p}{(2\pi)^3 2E_p} E_p (b^{r\dagger}(p) b^r(p) + c^{r\dagger}(p) c^r(p)),$$

good, $b^{r\dagger}(p)$ creates a spin 1/2 particle of positive energy, and $c^{r^\dagger}(p)$ creates a spin 1/2 particle of positive energy. The second term was re-ordered according to normal ordering – the terms originally work out to have the opposite order and the opposite sign. Fermionic statistics gives the sign above, upon normal ordering, but Bose statistics would have given the $c^{r\dagger}c^r$ term with a minus sign, leading to $H$ that is unbounded below. We need Fermionic statistics for spin 1/2 fields to get a healthy theory.

• Do perturbation theory as before, but account for Fermi statistics, e.g. $T(\psi(x_1)\psi(x_2)) = -T(\psi(x_2)\psi(x_1))$ and likewise for normal ordered products. Anytime Fermionic variables are exchanged, pick up a minus sign (and sometimes the additional term if the anti-commutator is non-zero). Consider in particular the propagator

$$\{\psi(x), \bar{\psi}(y)\} = (i\partial\!\!\!/_x + m)(D(x-y) - D(y-x)).$$

and the contraction

$$\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}.$$

Vanishes for spacelike separated points. The momentum space fermion propagator is

$$\frac{i}{p\!\!\!/ - m + i\epsilon}.$$