11/18/19 Lecture outline

• Continue from last time, Feynman path integral. As in the SHO QM example, we can compute field theory Green's functions via the generating functional (i.e. input a $J(x)$ and it outputs a number)

$$Z[J(x)] = \int [d\phi] \exp(i \int d^4x[\mathcal{L} + J(x)\phi(x)]).$$

Use it to compute Greens functions

$$G^{(n)}(x_1, \ldots x_n) \equiv \langle 0|T \prod_{i=1}^{n} \phi(x_i)|0\rangle / \langle 0|0\rangle = Z[J]^{-1} \prod_{j=1}^{n} \left(-i\frac{\delta}{\delta J(x_i)}\right) Z[J]\big|_{J=0}.$$

We can write this also as

$$Z[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^4x_i J(x_1) \ldots J(x_n) G^{(n)}(x_1, \ldots x_n).$$

where we now just remember to normalize $Z[J=0] = 1$. In general, use the generating functional $Z[J]$ to compute time ordered products, it reproduces Wick's theorem, Feynman diagrams, and thus S-matrix amplitudes (via LSZ).

Used

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(\frac{i}{\hbar}(\tfrac{1}{2}A_{ij}\phi_i\phi_i + J_i\phi_i)) = (2\pi i\hbar)^{N/2}(\det A)^{-1/2} \exp(-iA^{-1}_{ij}J_iJ_j/2\hbar)$$

E.g. for a free KG field, $A = (-\partial^2 - m^2 + i\epsilon)$, so For Klein-Gordon theory,

$$Z_0 = \int [d\phi] e^{iS/\hbar} \qquad S = \tfrac{1}{2} \int d^4x \phi(x)(-\partial^2 - m^2)\phi(x),$$

where we integrated by parts and dropped a surface term. This is completely analogous to our QM SHO example, simply replacing $\frac{d^2}{dt^2} + \omega^2 - i\epsilon$ there with $\partial^2 + m^2 - i\epsilon$ here – again, the $i\epsilon$ is to make the oscillating gaussian integral slightly damped. I.e. we should take $S = \frac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x)$, with $\epsilon > 0$, and then $\epsilon \to 0^+$. Note that the operator is $A \sim -\partial^2 - m^2 + i\epsilon$, which in momentum space is $p^2 - m^2 + i\epsilon$. Looks familiar: it's the Feynman $i\epsilon$ prescription, which here comes simply from ensuring that the integrals converge! This is why the path integral automatically gives the time ordering of the products. So

$$Z_0 = \mathrm{const}(\det(-\partial^2 - \mathrm{m}^2 + \mathrm{i}\epsilon))^{-1/2}.$$

The generating functional is

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4x d^4y J(x) D_F(x-y) J(y)), \tag{1}$$

with $(-\partial^2 - m^2 + i\epsilon)D_F(x-y) = i\delta^4(x-y)$ and $D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k^2-m^2+i\epsilon} = G^{(2)}(x,y)$.

• There is another nice combinatoric fact:

$$iW[J] \equiv \ln Z[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n} d^4x_i J(x_1)\dots J(x_n) G^{(n)}_{connected}(x_1,\dots x_n),$$

where $G^{(n)}_{connected}(x_1,\dots x_n)$ are the connected diagram Greens functions. For example, in the free field case we have $G_{connected}(x)^{(1)} = \frac{\hbar}{i}\frac{\delta}{\delta J(x)}W[J] = \langle\phi(x)\rangle_J = i\int d^4y D_F(x-y)J(y)$ and $G^{(2)}_{connected}(x_1,x_2) = \langle T\phi(x)\phi(y)\rangle_J - \langle\phi(x)\rangle_J\langle\phi(y)\rangle_J = D_F(x_1-x_2)$, and $G^{(n>2)}_{connected} = 0$. The LSZ result has those factors of $\prod_{i=1}^{n+m}(p_i^2-m_i^2)G^{(n+m)}$ to amputate the external legs, and that sets to zero the contributions of disconnected diagrams. So it suffices to consider only $G^{(n+m)}_{connected}$, and thus $W[J]$ is useful. $Z[J]$ has a formal relation to the partition function, and then $W[J]$ is related to free energy.

• Now let's consider an interacting theory. Notice that

$$\int[d\phi]\exp(\frac{i}{\hbar}[S_{free} + S_{int}[\phi] + \hbar\int d^4x J\phi]) = \exp[\frac{i}{\hbar}S_{int}[-i\frac{\delta}{\delta J}]) Z_{free}[J].$$

So

$$Z[J] = N\exp[\frac{i}{\hbar}S_{int}[-i\frac{\delta}{\delta J}]) Z_{free}[J], \tag{2}$$

where N is an irrelevant normalization factor (independent of J). The green's functions are then given by

$$\begin{aligned} G^{(n)}(x_1\dots x_n) &= \frac{\int[d\phi]\phi(x_1)\dots\phi(x_n)\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}{\int[d\phi]\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}, \\ &= \frac{1}{Z[J]}\prod_{j=1}^{n}\left(-i\hbar\frac{\delta}{\delta J(x_j)}\right)\cdot Z[J]\big|_{J=0}. \end{aligned}$$

(The denominator (in both lines) cancels off the vacuum bubble diagrams, which don't depend specifically on the Green's function.)

• Illustrate the above formulae, and relation to Feynman diagrams, e.g. $G^{(1)}$, $G^{(2)}$ and $G^{(4)}$ in $\lambda\phi^4$ theory. The functional integral accounts for all the Feynman diagrammer;

even symmetry factors etc. come out simply from the derivatives w.r.t. the sources, and the expanding the exponentials,

$$G^{(n)}(x_1,\dots x_n) = \frac{1}{Z[J]}\prod_{j=1}^{n}(-i\frac{\delta}{\delta J(x_j)})\sum_{N=1}^{\infty}\frac{1}{N!}\left(-i\frac{\lambda}{4!\hbar}\int d^4y(-i)^4\frac{\delta^4}{\delta J(y)^4}\right)^N Z_0[J]\big|_{J=0}.$$

etc. Consider, for example, the 4-point function $G^{(4)}(x_1,x_2,x_3,x_4) \equiv \langle T\phi(x_1)\dots\phi(x_4)\rangle/\langle 0|0\rangle$ in $\frac{\lambda_4}{4!}\phi^4$. So take 4-fuctional derivatives w.r.t. the source, at points $x_1\dots x_4$, i.e. $\delta/\delta J(x_1)\dots\delta/\delta J(x_4)$. The $\mathcal{O}(\lambda^0)$ term thus comes from expanding the exponent in (1) to quadratic order (4 J's), corresponding to the disconnected diagrams with two propagators. Each propagator ends on a point x_i. This is like the 4-point function in the SHO homework. Now consider the $\mathcal{O}(\lambda)$ contribution, coming from expanding out the interaction part of the exponent in (2) to $\mathcal{O}(\lambda)$. There are now 4 extra $\delta/\delta J(y)$, for a total of 8, so the contributing term comes from expanding the exponent in (1) to 4-th order, i.e. there are 4 propagators. This gives the connected term, along with several disconnected terms. Go through these terms and their combinatorics.

- Next topic: non-scalar fields (e.g. Fermions or spin 1 gauge fields).