11/4/19 Lecture outline

⋆ **Reading for the upcoming part: Coleman lecture notes pages 140-175.**

• Review Feynman rules for e.g. $\mathcal{H}_{int} = \frac{\lambda}{4!}\phi^4 + \frac{\lambda'}{6!}\phi^6$ and ask if there are any questions.

• Last time: brief introduction to a better description of QFT and perturbation theory (Lehmann, Symanzik, Zimmermann reduction formula). Define the true vacuum $|\Omega\rangle$ such that $H|\Omega\rangle = 0$, and $\langle\Omega|\Omega\rangle = 1$. The true vacuum of an interacting QFT is a complicated beast – it can be thought of roughly as a soup of particle-antiparticle states – it can not be solved for solved for exactly. (Progress: in classical mechanics, can solve 2 body problem exactly, but ≥ 3 body only approximately; in GR, can solve 1 body problem exactly, but ≥ 2 body only approximately; in QM can generally solve even only 1-body problem only approximately, but at least the 0-body problem is trivial; in QFT, even the 0-body problem is not exactly solvable.)

Define Green functions or correlation functions by

$$G^{(n)}(x_1, \dots x_n) = \langle\Omega|T\phi_H(x_1)\dots\phi_H(x_n)|\Omega\rangle,$$

where $\phi_H(x)$ are the full Heisenberg picture fields, using the full Hamiltonian.

Let's consider them in the free KG example. Find e.g. $G_0^{(2)}(x,y) = \hbar D_F(x-y)$, (where the subscript is to remind us it's the free theory), $G_0^{(4)} = G_0^{(2)}(x_1,x_2)G_0^{(2)}(x_3,x_4)+$ 2 permutations, etc.

Now show that in full generality (for any theory)

$$G^{(n)}(x_1 \dots x_n) = \frac{\langle 0|T\phi_{1I}(x_1)\dots\phi_{nI}(x_n)S|0\rangle}{\langle 0|S|0\rangle},$$

where $|0\rangle$ is the vacuum of the free theory, and ϕ_{iI} are interaction picture fields, and the S in the numerator and denominator gives the interaction-Hamiltonian time evolution from $-\infty$ to x_n, then from x_n to x_{n-1} etc and finally to $t = +\infty$. To show it, take $t_1 > t_2 \dots > t_n$ and put in factors of $U_I(t_a, t_b) = T\exp(-i\int_{t_a}^{t_b} dt H_I)$ to convert ϕ_I to ϕ_H, using $\phi_H(x_i) = U_I(t_i, 0)^\dagger \phi_I(x_i) U_I(t_i, 0)$.

Get $\langle 0|U_I(\infty, t_1)\phi_H(t_1)\dots\phi_H(t_n)U_I(t_n, -\infty)|0\rangle$, and U_I at ends can be replaced with full $U(t_1, t_2)$, since $H_0|0\rangle = 0$ anyway. Now use

$$\begin{aligned}
\langle\Psi|U(t,-\infty)|0\rangle &= \langle\Psi|U(t,-\infty)\left(|\Omega\rangle\langle\Omega| + \sum\int |n\rangle\langle n|\right)|0\rangle \\
&= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle + \lim_{t'\to-\infty}\sum\int e^{iE_n(t'-t)}\langle\Psi|n\rangle\langle n|0\rangle \\
&= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle
\end{aligned}$$

where 1 was inserted as a complete set of states, including the vacuum and single and multiparticle states, including integrating over their momenta, but the wildly oscillating factor kills all those terms. (Riemann-Lebesgue lemma: $\lim_{t\to\infty}\int d\omega f(\omega)e^{i\omega t}=0$ for nice $f(\omega)$) The result follows upon doing the same for the denominator.

The $\langle 0|S|0\rangle$ in the denominator eliminates the vacuum bubble diagrams. So we have

$$G^{(n)}(x_1,\ldots x_n)=\sum \text{Feynman graphs without vacuum bubbles.}$$

• Example: $G^{(2)}(x_1,x_2)$ in $\lambda\phi^4/4!$ theory is $G^{(2)}(x_1,x_2)=\Delta_F(x_1-x_2)+\frac{1}{6}(-i\lambda)^2\int d^4y_1\int d^4y_2\Delta_F(x_1-y_1)\Delta_F(y_1-y_2)^3\Delta_F(y_2-x_2)+\ldots$ where the $\frac{1}{6}$ is a **symmetry factor** from incompletely cancelling the two $1/4!$s, and is evident from the 3! permutation symmetry of the three internal propagators. Likewise, $G^{(4)}(x_1,x_2,x_3,x_4)$ in $\lambda\phi^4/4!$ theory. For each line from x to y, get a factor of $\Delta_F(x-y)$, and for each vertex at y get $-i\lambda\int d^4y$. Includes connected and disconnected diagrams. Disconnected ones will go away when computing (S-1)-matrix elements. In the LSZ formula (below) this is thanks to multiplying by factors of 0 associated with cancelling the external propagators, and those zeros eliminate the disconnected terms, as they have too few propagators to survive.

• It's often more convenient to work in momentum space,

$$\widetilde{G}^{(n)}(p_1,\ldots p_n)=\int\prod_{i=1}^{n}d^4x_i e^{-ip_ix_i}G^{(n)}(x_1\ldots x_n).$$

Then e.g. $\tilde{G}^{(2)}(p_1,p_2)=(2\pi)^4\delta^4(p_1+p_2)\left(\frac{i}{p_1^2-\mu^2+i\epsilon}\right)(1+\ldots)$. Similar to what we computed before to get S-matrix elements, but the external legs include their propagators, and the external momenta are not on-shell. There is a momentum conserving delta function with all momenta incoming.

• From Green functions $\widetilde{G}^{(n)}(p_1,\ldots,p_n)$, computed with external leg propagators, allowed to be off-shell, to S-matrix elements. Then

$$\langle k_{n+1}\ldots k_{n+m}|S-1|k_1k_2\ldots k_n\rangle=\prod_{j=1}^{n+m}\frac{k_j^2-m_j^2}{i\sqrt{Z_j}}\widetilde{G}^{(n+m}(k_1,k_2,\ldots,k_n,-k_{n+1},\ldots-k_{n+m}),$$

where the $(n+m)$ multiplicative factors are to amputate the external propagators. The Z_j "wavefunction renormalization" factors will be discussed without details below, and examples will be discussed and computed next quarter when you learn about renormalization of diagrams with loops. Because the $(n+m)$ external states are taken on shell, they have

$k_j^2 - m_j^2 \to 0$, so these multiplicative factors are $\sim 0^{n+m}$. These factors of zero cancel with factors of $1/0$ for diagrams with $n+m$ external propagators. Disconnected diagrams have fewer external propagators so they are set to zero by the 0^{n+m} multiplication. Consider for example $\widetilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ for 4 external mesons in our meson-nucleon toy model. The lowest order contribution is at $\mathcal{O}(g^0)$ and is

$$(2\pi)^4\delta^{(4)}(k_1+k_4)\frac{i}{k_1^2-\mu^2+i\epsilon}(2\pi)^4\delta^{(4)}(k_2+k_3)\frac{i}{k_2^2-\mu^2+i\epsilon} + 2 \text{ permutations}.$$

This is the -1 that we subtract in $S-1$, and indeed would not contribute to $2 \to 2$ scattering using the above formula, because it is set to zero by $\prod_{n=1}^4(k_n^2 - m_n^2)$ when the external momenta are put on shell. To get a non-zero result, need a $\widetilde{G}^{(4)}$ contribution with 4 external propagators, which we get e.g. at $\mathcal{O}(g^4)$ with an internal nucleon loop in the toy model with cubic interactions, or at tree-level in $\lambda\phi^4$, where it gives $i\mathcal{A}_{\phi^4} = -i\lambda + \mathcal{O}(\lambda^2)$.

• Account for bare vs full interacting fields. Let $|k\rangle$ be the physical one-meson state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}' - \vec{k})$. Then

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x}Z_\phi^{1/2}.$$

Can rescale the fields, s.t. $\langle k|\phi_R(x)|\Omega\rangle = e^{-ik\cdot x}$. The LSZ formula is:

$$\langle q_1 \dots q_n|S-1|k_1 \dots k_m\rangle = \prod_{a=1}^n \frac{q_a^2 - m_a^2}{i\sqrt{Z}} \prod_{b=1}^m \frac{k_b^2 - m_b^2}{i\sqrt{Z}} \widetilde{G}^{(n+m)}(-q_1, \dots - q_n, k_1, \dots k_m),$$

where the Green function is for the Heisenberg fields in the full interacting vacuum.

To derive the LSZ formula, consider wave packets, with some profile $F(\vec{k})$, and $f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k})e^{-ik\cdot x}$, where we define $k_0 = \sqrt{\vec{k}^2 + \mu^2}$, so $f(x)$ solves the KG equation. Now define (with the understanding that $\phi \to \phi_R$ is the rescaled Heisenberg picture field, shifted if necessary to eliminate any classical vacuum expectation value, i.e. if $\Omega|\phi|\Omega = v$ then shift $\phi \to \phi - v$ in what follows) the time-dependent and spatially independent operator

$$\phi^f(t) = i\int d^3\vec{x}(\phi(\vec{x},t)\partial_0 f(\vec{x},t) - f(\vec{x},t)\partial_0\phi(\vec{x},t)).$$

Note that, since $f(x)$ satisfies the KG equation, can show

$$i\int d^4x f(x)(\partial^2 + \mu^2)\phi(x) = -\int dt \frac{\partial}{\partial t}\phi^f(t) = -\phi^f(t)|_{-\infty}^{\infty} \equiv \phi^f_{in} - \phi^f_{out}.$$

Show that $\phi^f(t)$ makes **single particle** wave packets from the vacuum, $\langle k|\phi^f(t)|\Omega\rangle = F(\vec{k})$ (note that the two terms add). Can similarly show (because of a relative minus sign), that $\langle\Omega|\phi^f(t)|k\rangle = 0$ (instead, $\phi^{f\dagger}$ annihilates the single particle state). Let $|n\rangle$ be an n-particle momentum eigenstate: $P^\mu|n\rangle = P^\mu_n|n\rangle$. Then $\langle n|\phi^f(t)|\Omega\rangle = \frac{\omega_{p_n}+p_n^0}{2\omega_{p_n}}F(\vec{p}_n)e^{-i(\omega_{p_n}-p_n^0)t}\langle n|\phi(0)|\Omega\rangle$, where $\omega_{p_n} \equiv \sqrt{\vec{p}_n^2+\mu^2}$, which has $\omega_{p_n} < p_n^0$ for any multiparticle state. So for any state $\langle\psi|$ then $\lim_{t\to\pm\infty}\langle\psi|\phi^f(t)|\Omega\rangle = \langle\psi|f\rangle + 0$, where $|f\rangle \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k}F(k)\langle\psi|k\rangle$ is a one-particle state and the multiparticle states contributions sum to zero using the Riemann-Lebesgue lemma.

Make separated in states: $|f_n\rangle = \prod \phi^{f_n}(t_n)|\Omega\rangle$, and out states $\langle f_m| = \langle\Omega|\prod(\phi^{f_m})^\dagger(t_m)$, with $t_n \to -\infty$ and $t_m \to +\infty$. Then show

$$\langle f_m|S-1|f_n\rangle = \int\prod_n d^4x_n f_n(x_n)\prod_m d^4x_m f_m(x_m)^*\prod_r i(\partial_r^2+m_r^2)G(x_n,x_m).$$

Take $f_i(x) \to e^{-ik_ix_i}$ at the end. There are 2^{n+m} terms with different $t_i \to \pm\infty$. Can show that almost all cancel and we are left only with the n in-states and the m out-states. See the Coleman lectures for more details in the case of $2 \to 2$.

• Next topic: quantization via the Feynman path integral. In canonical quantization, time plays a special role: equal time commutation relations, time ordered products, etc. Feynman found a completely new description of QM, by intuitively thinking about double slit interference and realizing that empty space can be thought of as being filled with screens that are full of holes, so such interference and taking multiple paths is always there. The path integral generalizes immediately from QM to QFT, and for different types of fields. Unlike canonical quantization, it makes Lorentz and Poincare symmetry manifest, and also gives a way to define QFT beyond perturbation theory. The classical limit is clarified, as the stationary phase limit of an integral. Similar statements apply in optics. The path integral also helps to connect QFT with statistical physics, with the path integral analogous to the partition function.

First consider QM:

$$U(x_b,T;x_a,0) = \langle x_b|e^{-iHT/\hbar}|x_a\rangle = \int[dx(t)]e^{iS[x(t)]/\hbar}.$$

Integral can be broken into time slices, as way to define it.

E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2}\int\prod_{i=1}^{N-1}dx_i\exp[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i-x_{i-1})^2]$$

Where we take $\epsilon \to 0$ and $N \to \infty$, with $N\epsilon = T$ held fixed.

Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation):

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$. We'll see that this is related to the $i\epsilon$ that we saw in the Feynman propagator, which gave the T ordering.

After $n-1$ steps, get integral:

$$\left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2} \exp[\frac{m}{2\pi i\hbar n\epsilon}(x_n - x_0)^2].$$

So the final answer is

$$U(x_b, x_a; T) = \left(\frac{2\pi i\hbar T}{m}\right)^{-1/2} \exp[im(x_b - x_a)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions.

• Can derive the path integral from standard QM formulae, with operators, by introducing the time slices and a complete set of q and p eigenstates at each step.

$$\langle q', t'|q, t\rangle = \int\int \prod_{j=1}^{N} dq_j \langle q'|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle \ldots \langle q_1|e^{-iH\delta t}|q\rangle,$$

where we'll take $N \to \infty$ and $\delta t \to 0$, holding $t' - t \equiv N\delta t$ fixed. Note that even though $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]+\cdots}$, we're not going to have to worry about this for $\delta t \to 0$: $e^{-iH\delta t} = e^{-i\delta t p^2/2m} e^{-i\delta t V(q)} e^{\mathcal{O}(\delta t^2)}$. Now note

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int dp_1 \langle q_2|e^{-i\delta t p^2/2m}|p_1\rangle\langle p_1|e^{-iV(q)\delta t}|q_1\rangle,$$

$$= \int dp_1 e^{-iH(p_1,q_1)\delta t} e^{ip_1(q_2-q_1)}.$$

This leads to

$$\langle q', t'|q, t\rangle = \int [dq(t)][dp(t)] \exp(i \int_t^{t'} dt(p(t)\dot{q}(t) - H(p, q))),$$

and taking H quadratic in momentum and doing the p gaussian integral recovers the Feynman path integral.

• The same derivation as above leads to e.g.

$$\langle q_4, t_4|T\widehat{q}(t_3)\widehat{q}(t_2)|q_1, t_1\rangle = \int [dq(t)]q(t_3)q(t_2)e^{iS/\hbar},$$

where the integral is over all paths, with endpoints at (q_1, t_1) and (q_4, t_4).

A key point: the functional integral automatically accounts for time ordering! Note that the LHS above involves time ordered operators, while the RHS has a functional integral, which does not involve operators (so there is no time ordering). The fact that the time ordering comes out on the LHS is wonderful, since know that we'll need to have the time ordering for using Dyson's formula, or the LSZ formula, to compute quantum field theory amplitudes.

• The nice thing about the path integral is that it generalizes immediately to quantum fields, and for that matter to all types (scalars, fermions, gauge fields). E.g.

$$\langle \phi_b(\vec{x}, T)|e^{-iHT}|\phi_a(\vec{x}, 0)\rangle = \int [d\phi]e^{iS/\hbar} \qquad S = \int d^4x\mathcal{L}.$$

This is then used to compute Green's functions:

$$\langle \Omega|T \prod_{i=1}^{n} \phi_H(x_i)|\Omega\rangle = Z_0^{-1} \int [d\phi] \prod_{i=1}^{n} \phi(x_i) \exp(iS/\hbar),$$

with $Z_0 = \int [d\phi] \exp(iS/\hbar)$. Again, as noted above, the T ordering will be automatic.

• Now introduce **sources** for the fields as a trick to get the time order products from derivatives of a generating function (or functional).

• Consider QM with Hamiltonian $H(q, p)$, modified by introducing a source for q, $H \to H - J(t)q$. (We could also add a source for p, but don't bother doing so here.) Consider moreover replacing $H \to H(1-i\epsilon)$, with $\epsilon \to 0^+$, which has the effect of projecting on to the ground state at $t \to \pm\infty$. As mentioned, this'll be related to the $i\epsilon$ of the Feynman propagator. Consider the vacuum-to vacuum amplitude in the presence of the source,

$$\langle 0|0\rangle_J = \int [dq] \exp[i \int dt(L + J(t)q)/\hbar] \equiv Z[J(t)].$$

Once we compute $Z[J(t)]$ we can use it to compute arbitrary time-ordered expectation values. Indeed, $Z[J]$ is a generating functional[1] for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0|\prod_{j=1}^{n} Tq(t_j)|0\rangle = \prod_{j=1}^{n} \frac{1}{i}\frac{\delta}{\delta J(t_j)} Z[f]\Big|_{f=0},$$

where the time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture. We'll be interested in such generating functionals, and their generalization to quantum field theory (replacing $t \to (t, \vec{x})$).

• We'll want to compute amplitudes like

$$\frac{\langle 0|\prod_i Tq(t_i)|0\rangle_{J=0}}{\langle 0|0\rangle_{J=0}}$$

and for these the det A factor in the Gaussian integrals will cancel between the numerator and the denominator. This is related to the cancellation of vacuum bubble diagrams.

• Let's apply the above to compute the generating functional for the example of QM harmonic oscillator (scaling $m = 1$),

$$Z[J(t)] = \int [dq(t)] \exp(-\frac{i}{\hbar}\int dt \left[\tfrac{1}{2}q(t)(\frac{d^2}{dt^2} + \omega^2)q(t) - J(t)q(t)\right]).$$

This is analogous to the multi-dimenensional gaussian above, where i is replaced with the continuous label t, $\sum_i \to \int dt$ etc. and the matrix A_{ij} is replaced with the differential operator $A \to -(\frac{d^2}{dt^2} + \omega^2 - i\epsilon)$, where the $i\epsilon$ is to damp the gaussian, as mentioned above. Doing the gaussian gives a factor of $\sqrt{\det B}$ which we don't need to compute now because it'll cancel, and the exponent with the sources from completing the square, which is the term we want, so

$$\frac{\langle 0|0\rangle_J}{\langle 0|0\rangle_{J=0}} = \text{“}\exp[-i\tfrac{1}{2}A^{-1}_{ij}J_iJ_j/\hbar]\text{”} = \exp[-\tfrac{1}{2}\hbar\int dtdt' J(t)G(t-t')J(t')],$$

with $G(t)$ the Green's function for the oscillator, $(-\partial_t^2 - \omega^2 + i\epsilon)G(t) = i\delta(t)$,

$$G(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar}\frac{i \quad e^{-iEt/\hbar}}{E^2/\hbar^2 - \omega^2 + i\epsilon} = \frac{1}{2\omega}e^{-i\omega|t|}. \tag{1}$$

[1] Recall how functional derivatives work, e.g. $\frac{\delta}{\delta J(t)}J(t') = \delta(t-t')$.

The $i\epsilon$ here does the same thing as in the Feynman propagator: the pole at $E = \hbar\omega$ is shifted below the axis and that at $E = -\hbar\omega$ is shifted above. Equivalently, we can replace $E \to E(1 + i\epsilon)$, to tilt the integration contour below the $-\omega$ pole and above the $+\omega$ pole. Note then that $e^{-iEt/\hbar} \to e^{-iEt/\hbar}e^{Et\epsilon/\hbar}$, which projects on to the vacuum for $t \to \infty$ (the $i\epsilon$ projects on to the vacuum in the far future and also the far past).

For $t > 0$, the E contour is closed in the LHP and the residue is at $E = \hbar\omega$, while for $t < 0$ the contour is closed in the UHP, with residue at $E = -\hbar\omega$.

• Now that we know the generating functional, we can use it to compute time ordered expectation values via

$$\langle 0|T\prod_{i=1}^{n}\phi_H(t_i)|0\rangle/\langle 0|0\rangle = Z_0^{-1}\int [d\phi]\prod_{i=1}^{n}\phi(t_i)\exp(iS/\hbar) = Z_0^{-1}\prod_{i=1}^{n}\frac{\hbar}{i}\frac{\delta}{\delta J(t)}|_{J=0}.$$

with $Z_0 = \int [d\phi]\exp(iS/\hbar)$.

• On to QFT and the Klein-Gordon theory,

$$Z_0 = \int [d\phi]e^{iS/\hbar} \qquad S = \tfrac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2)\phi(x),$$

where we integrated by parts and dropped a surface term. This is completely analogous to our QM SHO example, simply replacing $\frac{d^2}{dt^2} + \omega^2 - i\epsilon$ there with $\partial^2 + m^2 - i\epsilon$ here – again, the $i\epsilon$ is to make the oscillating gaussian integral slightly damped. I.e. we should take $S = \frac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x)$, with $\epsilon > 0$, and then $\epsilon \to 0^+$. Note that the operator is $A \sim -\partial^2 - m^2 + i\epsilon$, which in momentum space is $p^2 - m^2 + i\epsilon$. Looks familiar: it's the Feynman $i\epsilon$ prescription, which you understood last quarter as needed to give correct causal structure of greens functions, here comes simply from ensuring that the integrals converge! This is why the path integral automatically gives the time ordering of the products. So

$$Z_0 = \mathrm{const}(\det(-\partial^2 - \mathrm{m}^2 + \mathrm{i}\epsilon))^{-1/2}.$$

As in the SHO QM example, we can compute field theory Green's functions via the generating functional

$$Z[J(x)] = \int [d\phi]\exp(i\int d^4x[\mathcal{L} + J(x)\phi(x)]).$$

This is a functional: input function $J(x)$ and it outputs a number. Use it to compute

$$\langle 0|T\prod_{i=1}^{n}\phi(x_i)|0\rangle/\langle 0|0\rangle = Z[J]^{-1}\prod_{j=1}^{n}\left(-i\frac{\delta}{\delta J(x_i)}\right)Z[J]\big|_{J=0}.$$

E.g. for the KG example, $A = (-\partial^2 - m^2 + i\epsilon)$, so the generating functional is

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4x d^4y J(x) D_F(x-y) J(y)), \tag{2}$$

with $(-\partial^2 - m^2 + i\epsilon)D_F(x-y) = i\delta^4(x-y)$ and $D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k^2-m^2+i\epsilon}$.

In general, use the generating functional $Z[J]$ to compute time ordered products, it reproduces Wick's theorem, Feynman diagrams, and thus S-matrix amplitudes (via LSZ).