9/30/19 Lecture outline

⋆ **Reading: Coleman lectures 0-2. Tong chapter 0-1.**

• In QM we learn $[x,p]=i\hbar$ whereas t is not quantized. It clearly violates relativity to treat x as an operator and t as not an operator. Also, in QM we learn that $\langle\psi|\psi\rangle=1$, where e.g. $|\psi\rangle$ is the state of a single electron. But in general particle number is not conserved. Quantum mechanics (QM) is an approximation to a deeper structure, quantum field theory (QFT), which is the subject of this class. The particles are actually ripples of fields, e.g. $\phi(t,\vec{x})$ and we quantize the fields ϕ themselves, with $\vec{x}$ a parameter, not a quantum operator. Actually, QM can be reinterpreted as a QFT in 0 space and 1 time dimension, by thinking about the positions $q^i(t)$ as the field operators.

• Non-relativistic QFT can be appropriate in condensed matter and other contexts. This class will focus on relativistic QFT. Conventions: $\hbar=c=1$, mostly minus metric $g_{\mu\nu}$, e.g. $\partial_\mu=(\partial_t,\vec{\nabla})$, $\partial_\mu\partial^\mu\equiv\partial^2=\partial_t^2-\nabla^2$.

• Q: "What goes wrong if we just do the single-particle S.E. with $H_{rel}=\sqrt{\vec{p}^2+m^2}$?" A: many things. One question is how to make sense of $H=\sqrt{\vec{P}^2+m^2}$, where $\vec{P}$ is an operator. How do you take a square-root of an operator? This question led Dirac to his equation, which describes fermions. Requires anti-matter to make sense of it. Can't have well-defined single-particle states. Here let's illustrate another issue: correlators outside the forward light-cone. Recall that any point outside the light-cone can be mapped by a Lorentz transformation to any other point, and that signals sent outside the light-cone can then be used to transmit information back in time, violating causality. As far as we're aware, that is illegal. Start with $|\psi(t=0)\rangle=|\vec{x}=0\rangle$. Compute, for $r>t$,

$$\begin{aligned}\langle\vec{x}|\psi(t)\rangle=\langle\vec{x}|e^{-iHt}|\vec{x}=0\rangle&=\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}e^{-i\sqrt{\vec{p}^2+m^2}t}\\&=-\frac{i}{(2\pi)^2r}\int_{-\infty}^{\infty}pdpe^{ipr}e^{-i\sqrt{p^2+m^2}t}\\&=-\frac{ie^{-mr}}{2\pi^2r}\int_m^\infty dzze^{-(z-m)r}\sinh(\sqrt{z^2-m^2}t)\end{aligned}.$$

The last step is by deforming the contour in the complex p plane. For $r>t$ we deform the contour into the UHP, and get contributions along the branch cut in the UHP, with $z=-ip$; you can verify that the contribution along the big semi-circle at infinity vanishes. The integral is positive, so non-vanishing outside the forward light cone: acausal, with causality recovered as an approximation for $r\gg m$. In QFT, antiparticles to the rescue!

The antiparticle contribution is added, and cancels the acausality. Must give up on purely single-particle states.

• Multiparticle states. E.g. $|\vec{k}_1, \vec{k}_2, \ldots \vec{k}_n\rangle$, with completeness

$$1 = |0\rangle\langle 0| + \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d^3\vec{k}_1}{(2\pi)^3} \cdots \frac{d^3\vec{k}}{(2\pi)^3} |\vec{k}_1 \ldots \vec{k}_n\rangle\langle \vec{k}_1 \ldots \vec{k}_n|.$$

Introduce a box for the moment, to make momenta discrete. Can then count how many excitations of each momenta. Fock space, as with the excitation level of the SHO, with creation and annihilation operators.

Recall the SHO: $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 = \omega(a^\dagger a + \frac{1}{2})$ (setting $m = 1$), and $[a, a^\dagger] = 1$. Ground state $|0\rangle$ s.t. $a|0\rangle = 0$, and $|n\rangle = c_n (a^\dagger)^n |0\rangle$.

Likewise, define $a(\vec{k})$ and $a^\dagger(\vec{k})$ s.t. $[a(\vec{k}), a^\dagger(\vec{k}')] = \delta_{\vec{k}\vec{k}'}$ for momenta in a box (will generalize to continuous, with Dirac delta functions). The multiparticle states are then $\prod_{i=1}^n a^\dagger(\vec{k}_i)|0\rangle$, and $H = \sum_{\vec{k}} \omega(\vec{k}) a^\dagger(\vec{k}) a(\vec{k})$.

• Classical and quantum particle mechanics, $L(q_a, \dot{q}_a, t)$, $p_a = \partial L/\partial \dot{q}_a$, $\dot{p}_a = \partial L/\partial q_a$, $H = \sum_a p_a \dot{q}_a - L$. Get quantum theory by replacing Poisson brackets with commutators, $[q_a(t), p_b(t)] = i\delta_{ab}$. Recall $O_H(t) = e^{iHt} O_S e^{-iHt}$ and $i\frac{d}{dt} O_H(t) = [O_H(t), H]$.

• Classical field theory. E.g. scalars $\phi_a(t, \vec{x})$, with $S = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a)$. Then $\Pi_a^\mu = \partial\mathcal{L}/\partial(\partial_\mu \phi_a)$, and E.L. eqns $\partial\mathcal{L}/\partial\phi_a = \partial_\mu \Pi_a^\mu$. Define $\Pi_a \equiv \Pi_a^0$. $H = \int d^3x (\Pi \dot{\phi}_a - \mathcal{L}) = \int d^3x \mathcal{H}$.

Example: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2)$, gives $\Pi = \dot{\phi}$ and $\dot{\Pi} = \nabla^2\phi - m^2\phi$, the Klein-Gordon equation: $(\partial^2 + m^2)\phi = 0$. Can't interpret ϕ as a probability wavefunction because of solutions $E = \pm\sqrt{\vec{p}^2 + m^2}$. But we'll see that the KG equation is fine as a field theory.