215a Homework exercises 6, Fall 2019, due Nov. 20

1. For a scalar field theory, with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, show that the EOM are satisfied, up to a contact term (this exercise is similar to one on an earlier HW set, but the idea is to write it out using $Z[J] = \int [d\phi] \exp(\frac{i}{\hbar}\int d^4x(\mathcal{L} + J(x)\phi(x))))$:

$$\langle T(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi(x) + V'(\phi(x)))\phi(y)\rangle = \alpha\delta(x-y).$$

   To do this problem, consider the functional integral and use the invariance of the functional integral under a change of variables. The change of variables is $\phi \to \phi + \epsilon f(x)$, where $f(x)$ is an arbitrary function of x, and ϵ is an infinitesimal parameter (drop terms of order ϵ^2 and higher). Derive in this way the above result and the coefficient α. (The Jacobian in the $[d\phi]$ integration measure doesn't contribute.)

2. This exercise introduces the basic ingredient for defining functional integrals for fermionic fields (e.g. the electron). Define anticommuting (a.k.a. "Grassmann") numbers by the multiplication rule $\theta\eta = -\eta\theta$, so $\theta^2 = 0$. A function of a single real Grassmann variable has a simple Taylor's expansion, $f(\theta) = A + B\theta$, where A and B are constants. Similarly, for a function of two real Grassmann variables, we have $f(\theta,\eta) = A + B\theta + C\eta + D\theta\eta$. Grassmann integration over a real Grassmann variable is defined by $\int d\theta = 0$ and $\int d\theta\theta = 1$ (Grassmann integration acts the same as differentiation.) Rather than working with real grassmann variables, let's package two real grassmann variables into a single complex grassmann variable: $\theta = (\theta_1 + i\theta_2)/\sqrt{2}$. Then e.g. $\int d\theta^* d\theta\theta\theta^* = 1$.
   Verify that $\int d\theta^* d\theta e^{-b\theta^*\theta} = b$, and more generally that

$$\prod_j \int d\theta_j^* d\theta_j e^{-(\theta^*, B\theta)} = \det B,$$

where $(\theta^*, B\theta) = \sum_{ij} B_{ij}\theta_i^*\theta_j$.