215a Homework exercises 3, Fall 2019, due Oct. 28

"Tong problem $n.m$" refers to exercise set n, problem m. Follow links from website.

1. Consider a real scalar field with ϕ^4 interaction: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$ in general D spacetime dimensions, so $S = \int d^Dx\mathcal{L}$.

(a) Write $\phi(x) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}2E_k}(a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ikx})$. We'll say that a and $a^\dagger$ annihilate and create "mesons." Recalling that $[.]$ denotes the mass dimension, and $[S] = 0$ for any D, write down $[\phi]$ and $[a(k)]$ and $[a(k)^\dagger]$ for all D.

(b) Define $\langle f|S|i\rangle = i\mathcal{A}_{fi}(2\pi)^D\delta^D(p_f - p_i)$. Suppose that $|i\rangle$ has n_i mesons and $\langle f|$ has n_f mesons. What is the mass dimension $[|i\rangle]$ and $\langle f|]$ and $[\mathcal{A}_{fi}]$?

(c) Find the amplitude (with D general) for the decay $\phi \to \phi+\phi+\phi$, to leading order in λ. Verify that your result is consistent with dimensional analysis and the above results. **Note:** the interaction term $:\mathcal{H}_I:$ should be taken to be normal ordered, so there is no contribution involving a loop.

2. Consider the theory $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \rho(x)\phi$, where $\rho(x^\mu)$ is an external source. [This exercise is taken from Peskin and Schroeder 4.1. You can find much of this worked out in the Coleman lecture notes, but please try to think about it and not peek / copy too much.]

(a) Give some words / explanation that the probability that the source creates no particles is given by

$$P(0) = |\langle 0|Te^{i\int d^4x\rho\phi_I}|0\rangle|^2,$$

where ϕ_I means interaction picture.

(b) Evaluate the above $P(0)$ to order ρ^2, thinking of ρ as a small perturbation. Show that $P(0) = 1 - \lambda + \mathcal{O}(\rho^4)$, where

$$\lambda \equiv \int \frac{d^3p}{(2\pi)^3(2E_p)}|\widetilde{\rho}(p)|^2,$$

where $\widetilde{\rho}(p) \equiv \int d^4xe^{ipx}\rho(x)$ (a tilde is often used to remind that it's a Fourier transform).

(c) Represent the term contributed in part (b) as a Feynman diagram. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that the series exponentiates so that it can be summed exactly: $P(0) = e^{-\lambda}$.

(d) Compute the probability that the source creates one particle of momentum k. Perform this first to $\mathcal{O}(\rho)$ and then exactly, using the trick of part (c).

(e) Show that the probability of producing n particles is given by the Poisson distribution:

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

(f) Note that $\sum_{n=0}^{\infty} P(n) = 1$ and $\langle N \rangle_{ave} \equiv \sum_{n=0}^{\infty} nP(n) = \lambda$, so λ is the expected number of created particles. Compute the mean square fluctuation $\langle (N - \langle N \rangle_{ave})^2 \rangle_{ave}$.

3. Tong 4.1 ($\lambda\phi^4$ question).