215a Homework exercises 2, Fall 2019, due Oct. 14

"Tong problem $n.m$" refers to exercise set n, problem m. Follow links from website.

1. Consider the KG theory $\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$.

   (a) Let $|p\rangle$ be the one-particle state $a^\dagger(p)|0\rangle$. Show that
$$\langle 0|\phi(x)|p\rangle = e^{-ip\cdot x}.$$

   (b) Using the expressions given in lecture for H and $\vec{P}$, show that
$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x).$$

2. In lecture, we used Fourier transforms to construct Greens functions for the Klein-Gordon equation. In this exercise, you will explicitly verify that
$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2\right)\langle 0|T(\phi(x)\phi(y))|0\rangle = C\delta^4(x-y),$$

and determine the constant C. Don't use the Fourier transforms from lecture. Instead just use the KG field equation for $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$, the equal time commutation relations, and the definition of the time ordered product T. Hint: use the definition of T involving $\theta(x^0 - y^0)$ and $\theta(y^0 - x^0)$, and the fact that the derivative of the theta function gives a delta function.

This exercise illustrates that equations of motion, which are operator equations, don't necessarily give zero in time ordered correlation functions – instead, it can give contact interactions when the operator points coincide. We'll soon see similar effects with current conservation laws, which can also have specific contact interactions when $\partial_\mu j^\mu(x)$ is in a correlation function (these are called Ward identities).

3. Tong problem set 2, exercise 9 (i.e. explicitly verify Wick's theorem for the case of three scalar field operators).

4. This is a warm-up or review of the path integral description of quantum mechanics (and Gaussian integrals). Consider the propagator of 1d QM: $K(x_2, t_2; x_1, t_1) \equiv \langle x_2|U(t_2, t_1)|x_1\rangle$, i.e. the probability amplitude to go from initial position x_1 at time t_1 to final position x_2 at time t_2. Suppose that $H = \frac{1}{2m}p^2 + V(x)$ is time independent, so $U = e^{-iHT/\hbar}$ with $T \equiv t_2 - t_1$.

   (a) Compute K_{free} for the case of a free particle $V = 0$ by inserting $1 = \int \frac{dp}{2\pi}|p\rangle\langle p|$ and doing the Gaussian integral (recall that $\int dx e^{-\lambda x^2} = \sqrt{\pi/\lambda}$ and for the moment just assume that this works even if λ is imaginary).

   (b) Verify that $K_{free} = f(T)e^{iS_{cl}/\hbar}$ where S_{cl} is the classical action for the path.

   (c) Argue (for any $V(x)$) that $K(x_3, t_3; x_1, t_1) = \int dx_2 K(x_3, t_3; x_2, t_2)K(x_2, t_2; x_1, t_1)$ assuming that $t_3 \geq t_2 \geq t_1$. Explicitly verify this for K_{free} by doing the $\int dx_2$ integral.