215a Homework exercises 1, Fall 2019, due Oct. 7

"Tong problem $n.m$" refers to exercise set n, problem m. Follow links from website.

1. Consider a complex scalar field with

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{2}(\phi^\dagger \phi)^2$$

Note that the theory is invariant under $\phi \to e^{i\alpha}\phi$, with α constant (i.e. a global symmetry). Derive the associated Noether current and verify that it is conserved, using the field equations satisfied by ϕ.

2. Tong problem set 1, exercise 8.

3. Tong problem set 1, exercise 9. Please call the scaling dimension Δ instead of D, and another notation is to write $[\phi]$, where the square brackets means the scaling mass dimension of ϕ, e.g. in $\hbar = c = 1$ units $[E] = [m] = [p] = [1/t] = [1/L] = 1$, and $[S] = [\hbar] = 0$ in any spacetime dimension.

4. Consider a complex scalar field with

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$

Define

$$\phi(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{-ik\cdot x} + b^\dagger(k)e^{ik\cdot x} \right).$$

$$\phi^\dagger(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)^\dagger e^{ik\cdot x} + b(k)e^{-ik\cdot x} \right).$$

(a) Find the units $[a(k)]$ and $[b(k)]$.

(b) Find the conjugate coordinate $\Pi(x)$ to $\phi(x)$. Also find the units $[\Pi(x)]$.

(c) Impose the canonical equal-time commutation relation $[\phi(\vec{x},t), \Pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$ and show this implies that $[a(k), a^\dagger(k')] = [b(k), b^\dagger(k)] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}')$, with all other commutators vanishing. Verify that these are compatible with $[a(k)]$ and $[b(k)]$.

(d) Recall from question 1 that there is a conserved current, $j^\mu(x)$ with $\partial_\mu j^\mu = 0$, corresponding to the $\phi \to e^{i\alpha}\phi$ symmetry. Write the corresponding charge $Q = \int d^3x j^0$ as $Q = \int d^3k \ldots$, where $\ldots$ is in terms of things like $a(k)$ and $b(k)$. Write Q as a normal ordered expression, so $Q|0\rangle = 0$. Verify that Q is dimensionless, i.e. that $[Q] = 0$.

(e) Verify that $a^\dagger(k)|0\rangle$ and $b^\dagger(k)|0\rangle$ are eigenstates of Q. What are their eigenvalues?