Physics 105a, Ken Intriligator lecture 9, October 26, 2017

• Recall periodic $f(t+T)=f(t)$ has a Fourier expansion

$$f(t)=\sum_{n=-\infty}^{\infty}\tilde{f}_n e^{-2\pi int/T}=a_0+\sum_{n=1}^{\infty}(a_n\cos(2\pi nt/T)+b_n\sin(2\pi nt/T)).$$

With

$$\tilde{f}_n=\int_{t_0}^{T+t_0}\frac{dt}{T}f(t)e^{2\pi int/T}.$$

Or equivalently

$$a_0=\int_{t_0}^{T+t_0}\frac{dt}{T}f(t),\quad a_{m>0}=2\int_{t_0}^{T+t_0}\frac{dt}{T}f(t)\cos(2\pi mt/T),\quad b_{m>0}=2\int_{t_0}^{T+t_0}\frac{dt}{T}f(t)\sin(2\pi mt/T),$$

Note that, for real $f(t)$, $\tilde{f}_n=\tilde{f}^*_{-n}$. The exponential and sin and cos forms are related by $\tilde{f}_0=a_0$, $\tilde{f}_{n>0}=\frac{1}{2}(a_n+ib_n)$, $\tilde{f}_{n<0}=\frac{1}{2}(a_{-n}-ib_{-n})$.

• Aside: can write it in QM notation (which is a convenient way to write basis vectors and their inner products): $|f\rangle=\sum_n\tilde{f}_n|n\rangle$, where $\tilde{f}_n=\langle n|f\rangle$ and $\langle n|m\rangle=\delta_{n,m}$. Now $\langle t|n\rangle=e^{-2\pi int/T}/\sqrt{T}$ are the normalized basis vectors in t-space and the notation encodes the fact that the inner product involves complex conjugation with $\langle n|t\rangle=\langle t|n\rangle^*$ generally complex. The orthogonality and completeness relations can be written as $\int_{t_0}^{T+t_0}dt|t\rangle\langle t|=\mathbf{1}$ and $\sum_n|n\rangle\langle n|=\mathbf{1}$, the unit operator (think of it as a matrix with 1's on the diagonals and 0's off-diagonal).

• Approximate the function by keeping up to M terms in the Fourier sum:

$$f_{approx}(t,M)=\sum_{n=-M}^{M}\tilde{f}_n e^{-2\pi int/T}=a_0+\sum_{n=1}^{M}(a_n\cos(2\pi nt/T)+b_n\sin(2\pi nt/T)).$$

We saw examples using Mathematica, comparing to $f(t)$ and plotting $f_{error}(t,M)=f(t)-f_{appox}(t,M)$.

• We discussed the square and triangle wave examples, and briefly illustrated the Gibbs phenomenon with Mathematica. Continue with that. Recall that the square wave had $a_n=0$ since it was defined to be an odd function and $b_{2m}=0$ and $b_{2m+1}=4/(2m+1)\pi$. So

$$f_{approx}(t,M)=\frac{4}{\pi}\sum_{n=odd=1}^{M}\frac{1}{n}\sin(2\pi nt/T).$$

Because the function is discontinuous, $f_{error}(t,M)=f(t)-f_{appox}(t,M)$ does not go to zero for $M\to\infty$. To see that, do some substitutions such that the $M\to\infty$ limit of

$f_{approx}(t,M)$ will look nicer: substitute $\tau = Mt/T$ and $s = n/M$ to get, with $\Delta s = 2/M \to 0$,

$$f_{approx} \to \sum_{s=odd\Delta s/2}^{1} \Delta s s^{-1} \sin(2\pi s\tau) \to \int_0^1 \frac{1}{s}\sin(2\pi s\tau)ds = \frac{2}{\pi} SinIntegral[2\pi\tau].$$

Consider derivative and where it vanishes. The maximum overshoot is at $\tau = 1/2$. Continue with mathematica.

• Solving the forced, damped, SHO for general periodic functions via Fourier series. Write $x_p(t) = \sum_n \tilde{x}_n e^{-in\omega_1 t}$ and $f(t) = \sum_n \tilde{f}_n e^{-in\omega_1 t}$, where $\omega_1 = 2\pi/T$. Relate $\tilde{x}_n$ to $\tilde{f}_n$ as in last week's discussion of the forced SHO:

$$\tilde{x}_n = \tilde{f}_n/(-(n\omega_1)^2 - in\gamma n\omega_1 + \omega_0^2).$$

As we saw last week, this has a magnitude and a phase, and the phase gives a lag between the forcing and the response. For high frequency modes, $n \gg 1$, find $\tilde{x}_n \approx -\tilde{f}_n/(n\omega_1)^2$, so the response falls off with two more powers of $1/n$ than the forcing. For small n, get $\tilde{x}_n \approx \tilde{f}_n/\omega_0^2$, which is simply $x_p(t) \approx -f(t)/k$ for the slowly varying terms in $f(t)$. When the denominator in the relation between $\tilde{x}$ and $\tilde{f}$ is small, those modes are near resonance and thus amplified relative to the others.

• Fourier series for functions on a finite interval, via periodic extension. Examples with $f^{(p)}(t+T) = f(t)$, where $T = \Delta t$ of the original interval. Such functions are discontinuous generally, so the Fourier sums do not converge uniformly to the original function: their F.T. displays Gibbs phenomena at the boundary of the interval. Show in mathematica.

An alternative is to define the function outside of the interval to have period $2T$ and make it an even function, reflecting it around one endpoint. Then the function will be continuous everywhere, though with generally discontinuous first derivative. The Fourier transforms approximate the function better than the original ones. Examples with $f^{(e)}(t) = f^{(e)}(t+2T)$, where the function in the range $t \in [t_0 - T, t_0]$ is defined via $f^{(e)}(t) = f(2t_0 - T)$, making it even upon reflection at t_0, so it has half as many jumps. Alternatively, take $f^{(o)}(t) = f^{(o)}(t+2T)$ with $f^{(o)}(t) = -f(2t_0 - t)$ in the range $[t_0 - T, t_0]$, so get sin's instead of cos's in expansion around t_0. Note the better falloff for the odd extension w.r.t. the mode number n, for fixed function $f(t)$, as compared with the original or the even extension.

• Solving boundary value problems using Fourier series, e.g. for the shaken spring demo. Examples with $\phi(x)$ and $\partial_x^2\phi = -\rho(x)$ with Dirichlet (fixed value) and boundary, say $\phi(0) = \phi_0$ and $\phi(L) = \phi_1$, do expansion in $\sin(n\pi x/L)$. For Neumann boundary conditions the derivatives at the endpoints are instead specified, and then we can instead expand in $\cos(n\pi x/L)$.