Physics 105a, Ken Intriligator lecture 7, October 19, 2017

• Last time: $L\psi(t) = f(t)$, where L is a linear differential operator:

$$L \equiv \sum_{n=0}^{N} a_n \frac{d^n}{dt^n}.$$

The solution is given by a superposition of homogenous and particular solutions: $\psi = \psi_h + \psi_p$, where $L\psi_h = 0$, and $L\psi_p = f(t)$. Find $\psi_h = Re(\sum_{n=1}^{N} c_n e^{s_n t})$, where c_n are the N expected constants of integration and s_n are the solutions of the polynomial equation $\sum_{n=0}^{N} a_n s^n = 0$. If solutions are degenerate (coinciding roots) get powers of t as different solutions. Example: damped harmonic oscillator, get $s^2 + \gamma s + \omega_0^2 = 0$, so $s = \frac{1}{2}(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2})$. Note that we are solving this over the complex values of s, so we always get two solutions. If $\gamma^2 > 4\omega_0^2$, this is the over damped case, and both solutions have real s, i.e. the solutions are exponentials in t. For $\gamma > 0$, both solutions are exponentially decaying for $t \to +\infty$. If $\gamma^2 < 4\omega_0^2$ (under-damped case) the solutions are $s = -\frac{1}{2}\gamma \pm i\omega_1$, where $\omega_1 = \sqrt{\omega_0^2 - \frac{1}{4}\gamma}$, i.e. we get $\psi_h(t) = e^{-\frac{1}{2}\gamma t}(A\cos(\omega_1 t) + B\sin(\omega_1 t))$. If $\gamma^2 = 4\omega_0^2$ (critically damped case), there seems to be only one solution. In the critically damped SHO case, get $\psi_h(t) = e^{-\frac{1}{2}\gamma t}(A + Bt)$.

We will later study some general methods to determine the particular solution to $L\psi_p(t) = f(t)$ for general $f(t)$. For the moment, consider the case where $f(t) = Re f_0 e^{-i\omega t}$ and note that the particular solution can be found by an obvious guess: $\psi_p(t) = Re C e^{-i\omega t}$. Plugging in get $\sum_n a_n(-i\omega)^n C = f_0$, which we can solve for C.

For example, for the damped SHO we get $C = f_0(-\omega^2 - i\omega\gamma + \omega_0^2)^{-1} = f_0(\omega_0^2 - \omega^2 + i\omega\gamma)((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)^{-1}$. The magnitude is largest for $\omega_{res} = \sqrt{\omega_0^2 - \frac{1}{2}\gamma^2}$. The imaginary part is a phase shift lag between the forcing motion and the oscillation.

• Now consider differential equations with boundary value conditions. For example, consider $L\psi = 0$ with $L = \frac{d^2}{dt^2} - \omega^2$, or $L = \frac{d^2}{dx^2} - k^2$. These are the same equation mathematically, and physically we use different names because sometimes we have such equations in time, and other times we have such equations in space. Suppose e.g. we try to specify $\psi(t) = \psi(t + T)$; this only has a solution if $T = 2\pi/\omega$. Likewise if we try to specify $\psi(x) = \psi(x + \lambda)$, there is only a solution if $k = 2\pi/\lambda$. Such equations and boundary conditions e.g. give the allowed frequencies or wavenumber of waves on strings, or musical instruments, or the allowed energy levels in quantum mechanics. This illustrates that boundary value problems do not always have a solution - it depends on if the boundary conditions are compatible, like trying to fit different pieces of a jigsaw puzzle together.

Examples from waves on string, and musical instruments, with either zero displacement or zero slope at ends. Plot examples using mathematica.

• Numerical solutions of boundary value problems by the shooting method. E.g. $\frac{d^2y}{dt^2} = f(y, \dot{y}, t)$ with $y(t_0) = y_0$ and $y(t_1) = y_1$. Guess $\dot{y}(t_0)$ to try to get $y(t_1) = y_1$, and adjust as needed. Example from Dubin 1.70.

• Think about solving $L\psi = f$ in analogy with a matrix equation, where L is a matrix and ψ and f are column vectors. Then ψ_h is in the nullspace of L, and $\psi_p = L^{-1} f$, where L^{-1} is the inverse in the directions orthogonal to the nullspace. We are literally doing this when we numerically solve the differential equation by making the coordinate a lattice of points, as in the Euler's method example. Replace e.g. $\frac{d}{dt} \to L_{n,m} = (\delta_{n,m} - \delta_{n-1,m})/\Delta t$.

• Example from Dubin 1.6.3: $L = \frac{d}{dt} + u_0(t)$ via Euler's method. Consider $u_0(t) = 1$.

• Example from Dubin 1.4.4: Predictor-Corrector Method of order 2.