Physics 105a, Ken Intriligator lecture 6, October 13, 2017

• Phase space motion from Hamilton's equations: $H(x,p,t)$ with $\dot{x} = \partial_p H$ and $\dot{p} = -\partial_x H$. SHO example. Discuss $\dot{H}$ vs $\partial_t H$ and show that $\dot{H} = 0$ if $\partial_t H = 0$: this is conservation of energy if the system does not explicitly depend on t. Get $dH = 0$, so the flow is along surfaces of constant H. You will learn more about this in physics 110.

• Consider the differential equation $\frac{d^2\psi}{dt^2} + \gamma\frac{d\psi}{dt} + \omega_0^2\psi = f(t)$, where γ and ω_0^2 are constants and $f(t)$ is some given function. This equation arises in the forced, damped, harmonic oscillator, where $\psi(t) = x(t)$. It also arises in a circuit with an inductor L, resistor R, and capacitor C, where $\psi(t) = q(t)$ is the charge on the capacitor, $\gamma = R/L$, $\omega_0^2 = 1/LC$, and $f(t) = v(t)/L$.

More generally, consider a *linear* N-th order differential equation $L\psi(t) = f(t)$, where L is a linear differential operator:

$$L \equiv \sum_{n=0}^{N} a_n \frac{d^n}{dt^n}.$$

The solution is given by a superposition of homogenous and particular solutions: $\psi = \psi_h + \psi_p$, where $L\psi_h = 0$, and $L\psi_p = f(t)$. Let's first consider the homogenous equation, i.e. for now we set $f(t) = 0$. Since we effectively need to integrate N times, there should be N undetermined constants, which can be fully determined if N initial conditions are specified. When solving $\vec{F} = m\frac{d^2\vec{x}}{dt^2}$, we have $N = 2$, and the motion can be specified by the initial position and initial velocity.

We will soon discuss boundary value problems, where instead of specifying values of ψ and its derivatives at one location of the variable, instead values of the solution (and/or its derivative) are specified at different locations of the variable.

• Solve $L\psi = 0$ by superposition (since it's linear): $\psi = Re(\sum_{n=1}^{N} c_n e^{s_n t})$, where c_n are the N expected constants of integration and s_n are found by computing Le^{st} and setting it to zero, i.e. they are the solutions of the polynomial equation $\sum_{n=0}^{N} a_n s^n = 0$.

• Example: damped harmonic oscillator, get $s^2 + \gamma s + \omega_0^2 = 0$, so $s = \frac{1}{2}(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2})$. Note that we are solving this over the complex values of s, so we always get two solutions. If $\gamma^2 > 4\omega_0^2$, this is the over damped case, and both solutions have real s, i.e. the solutions are exponentials in t. For $\gamma > 0$, both solutions are exponentially decaying for $t \to +\infty$. If $\gamma^2 < 4\omega_0^2$ (under-damped case) the solutions are $s = -\frac{1}{2}\gamma \pm i\omega_1$, where $\omega_1 = \sqrt{\omega_0^2 - \frac{1}{4}\gamma}$, i.e. we get $\psi_h(t) = e^{-\frac{1}{2}\gamma t}(A\cos(\omega_1 t) + B\sin(\omega_1 t))$. If $\gamma^2 = 4\omega_0^2$ (critically damped case), there seems to be only one solution. But there must be two.

To get insight into the critically damped case, consider as a warm-up the differential equation $\frac{d^N\psi}{dt^N} = 0$. If we plug in e^{st}, we get $s^N = 0$, which seems to suggest $\psi = c$ a constant as the only solution. There must be N independent solutions, and indeed there are: $\psi = \sum_{n=0}^{N-1} c_n t^n$. We can sort of understand them from e^{st} with $s^N = 0$: Taylor expand and keeping powers of (st) up to $s^N = 0$ gives the correct solution. Likewise, whenever the roots of the polynomial equation for s coincide, we get solutions like e^{st} multiplied by powers of t.

In particular, in the critically damped SHO case, get $\psi(t) = e^{-\frac{1}{2}\gamma t}(A + Bt)$.

• We will later study some general methods to determine the particular solution to $L\psi_p(t) = f(t)$ for general $f(t)$. For the moment, consider the case where $f(t) = Ref_0 e^{-i\omega t}$ and note that the particular solution can be found by an obvious guess: $\psi_p(t) = ReCe^{-i\omega t}$. Plugging in get $\sum_n a_n(-i\omega)^n C = f_0$, which we can solve for C.

For example, for the damped SHO we get $C = f_0(-\omega^2 - i\omega\gamma + \omega_0^2)^{-1} = f_0(\omega_0^2 - \omega^2 + i\omega\gamma)((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)^{-1}$. The magnitude is largest for $\omega_{res} = \sqrt{\omega_0^2 - \frac{1}{2}\gamma^2}$. The imaginary part is a phase shift lag between the forcing motion and the oscillation.

• Now consider differential equations with boundary value conditions. For example, consider $L\psi = 0$ with $L = \frac{d^2}{dt^2} - \omega^2$, or $L = \frac{d^2}{dx^2} - k^2$. These are the same equation mathematically, and physically we use different names because sometimes we have such equations in time, and other times we have such equations in space. Suppose e.g. we try to specify $\psi(t) = \psi(t+T)$; this only has a solution if $T = 2\pi/\omega$. Likewise if we try to specify $\psi(x) = \psi(x+\lambda)$, there is only a solution if $k = 2\pi/\lambda$. Such equations and boundary conditions e.g. give the allowed frequencies or wavenumber of waves on strings, or musical instruments, or the allowed energy levels in quantum mechanics. This illustrates that boundary value problems do not always have a solution - it depends on if the boundary conditions are compatible, like trying to fit different pieces of a jigsaw puzzle together.

• Numerical solutions of boundary value problems by the shooting method. E.g. $\frac{d^2y}{dt^2} = f(y, \dot{y}, t)$ with $y(t_0) = y_0$ and $y(t_1) = y_1$. Guess $\dot{y}(t_0)$ to try to get $y(t_1) = y_1$, and adjust as needed. Example from Dubin 1.70.

• Think about solving $L\psi = f$ in analogy with a matrix equation, where L is a matrix and ψ and f are column vectors. Then ψ_h is in the nullspace of L, and $\psi_p = L^{-1}f$, where L^{-1} is the inverse in the directions orthogonal to the nullspace. We are literally doing this when we numerically solve the differential equation by making the coordinate a lattice of points, as in the Euler's method example. Replace e.g. $\frac{d}{dt} \to L_{n,m} = (\delta_{n,m} - \delta_{n-1,m})/\Delta t$.

• Example from Dubin 1.6.3: $L = \frac{d}{dt} + u_0(t)$ via Euler's method. Consider $u_0(t) = 1$.