Physics 105a, Ken Intriligator lecture 3, October 5, 2017

• Summary from where we ended last time: we can take any function $f(x)$ and try to promote it to a function $f(z = x + iy)$ on the complex plane by just replacing $x \to z$. In order for the derivatives of $f(z)$ to be well defined, we need to get the same answer if we take $dz = dx$ or $dz = idy$, and this gives the Cauchy Riemann equations. If $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, the term with $n = -1$ is special: that term is called a pole, and its coefficient, a_n, is called the residue of the function at that pole. More generally, there is a pole near some point z_0 if $f(z \approx z_0) = (coeff)(z - z_0)^{-1} + \ldots$, where the coefficient is called the residue of the pole at z_0, and none of the other terms in $\ldots$ matter. Aside: if $f(z) = (z - z_0)^p + \ldots$ for p not an integer, then there is a "cut" starting at z_0, e.g. $\sqrt{z - z_0}$. Cauchy's theorem says that $\oint_C f(z)dz$ is $2\pi i \sum_{poles} residues$, where the poles are those inside of C (and we need to avoid any cuts or take care when crossing it, or sometimes it's useful to hug around either side of a cut and account for the difference). There are many applications to physics. We will only scratch the surface of these methods in this class.

• Plot in mathematica (using VectorPlot of the real and imaginary parts) e.g. $f(z) = z^n$ for various n, $f(z) = \bar{z}$ (not analytic), $z^{1/2}$, $z^{1/3}$.

• Note that the CR equations imply that $\Re f(z)$ and $\Im f(z)$ are solutions of the 2d Laplace equations; examples. Useful for e.g. some electrostatics problems. Verify it for some examples using Mathematica.

• Discuss $\oint_C dz/z = 2\pi i$ in terms of $z^{-1} = \partial_z \log z$ and the behavior of $\log z$ in the complex plane.

• Write $f(z)dz$ in terms of real and imaginary parts, and then as $(\vec{F} \cdot d\ell, (d\vec{\ell} \times \vec{F})$, with $\vec{F} = (u, -v)$, and note that the CR equations imply that $\vec{F}$ has no divergence or curl, clarifying why $\oint f(z)dz$ is "almost zero", up to the effects from the poles. Indeed, the poles are places where singularities of the derivatives of a certain type. This is related to the fact that $\log(z - z_0)$ is a Green's function for the 2d Laplacian. We will discuss Green's functions later.

• Continue with example of $\int_0^\infty dx(1 + x^2)^{-1} = \pi/2$ and show that one gets the same answer if C is closed instead in the lower half plane, accounting for the sign convention.

• Other examples of evaluating integrals by Cauchy's theorem. $\int_0^\pi d\theta/(a + b\cos\theta) = \pi/\sqrt{a^2 - b^2}$,

• Residues and poles of $\pi/\sin(\pi z)$ and $\pi\cos(\pi z)/\sin(\pi z)$ and applications of Cauchy's theorem to evaluate some sums, $\sum_{n=1}^{\infty} f(n)$. Examples.

• Example: consider an L, R circuit, driven by source $V(t) = A\int e^{i\omega t}d\omega/2\pi$; this source corresponds to a voltage spike at time $t = 0$. Find $I(t) = A\int(R + i\omega L)^{-1}d\omega/2\pi$. Discuss where to close the contour and get $I(t < 0) = 0$ and $I(t > 0) = (A/L)e^{-Rt/L}$ – makes sense.

• Gamma function $\Gamma(z)$; give integral definition and type it into mathematica, $\Gamma(z + 1) = z\Gamma(z)$ and relation to factorial. Poles at $x = 0$ and negative integers. Check with mathematica. Also $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$.

• Gaussian integral, including in multi-dimensions. Normalization of the normal distribution. Relation to spherical integrals and solid angles.

• Suppose that we want to solve the ODE $\frac{d^2x}{dt^2} = f(x, \dot{x})$, where f is some given function, e.g. $f = -\omega_0^2 x - \gamma v$ for the case of a damped SHO. Note that we are here taking $f(x, \dot{x})$ to not depend explicitly on t. Plot $(x, \dot{x}, t)$ and discuss projection of motion onto the $(x, \dot{x})$ plane. Discuss example in cell 1.6 of Chapter1.nb. It is often useful to use p instead of $\dot{x}$ (in simple cases, this is just a rescaling as $p = m\dot{x}$). Plot phase space motion for the solution of the undamped SHO.

• Non- dissipative systems have conserved energy and the flow in the (x, v) plane has zero divergence. Hence the area in phase space is constant in time.

• Hamiltonian flows: $H(x, p, t)$ with $\dot{x} = \partial_p H$ and $\dot{p} = -\partial_x H$. Discuss $\dot{H}$ vs $\partial_t H$ and show that $\dot{H} = 0$ if $\partial_t H = 0$: this is conservation of energy if the system does not explicitly depend on t. You will learn more about this in physics 110.