Physics 105a, Ken Intriligator lecture 18, Dec 5, 2017

• Continue from last time: $\nabla^2\phi = -\rho$, Or the wave equation in more space dimensions, $\nabla^2\psi - \frac{1}{c^2}\partial_t^2\psi = 0$. Or the heat equation in more space dimensions, $\partial_t T = \chi\nabla^2 T$. Solutions of the Laplace equation with N or D boundary conditions are unique.

2d rectangular: $\phi(x,y) = X(x)Y(y)$. Get $X'' = -k_1^2 X$ and $Y'' = -k_2^2 Y$ with $k_1^2 + k_2^2 = 0$, so the solutions are with $X(x) = C_1 e^{\kappa x} + C_2 e^{-\kappa x}$ and $Y(y) = C_3 e^{i\kappa y} + C_4 e^{-i\kappa y}$, where κ is real or imaginary depending on the BCs. The solutions are sin and cos in one directions, and sinhs and coshs in the other. The direction with oscillating solutions leads to a single n sum.

Example: suppose that BCs are $\phi(x,0) = \phi(y,0) = \phi(x,b) = 0$ and $\phi(a,y) = \phi_A(y)$. Then $Y(0) = Y(b) = 0$ which requires that the oscillatory direction is y, and the exponential direction is x:

$$\phi(x,y) = \sum_n A_n \sinh(n\pi x/b)\sin(n\pi y/b), \qquad A_n \sinh(n\pi a/b) = \frac{2}{b}\int_0^b \phi_A(y)\sin(n\pi y/b)dy.$$

• Now consider 3d rectangular, $\phi(x,y,z) = X(x)Y(y)Z(z)$ get $X'' = -k_1^2 X$ and $Y'' = -k_2^2 Y$ and $Z'' = -k_3^2 Z$ with $k_1^2 + k_2^2 + k_3^2 = 0$, so at least one has to be real and at least one has to be imaginary, i.e. there is at least one oscillating direction and at least one exponential direction. Again, solve for which is which based on the BCs.

Example: conducting wire with insulated sides: $\phi(x,y,0) = 0$, $\phi(x,y,L) = V_0(x,y)$, with ends at $x = 0$, $x = a$, $y = 0$, and $y = b$ insulated. Ohms law gives $\vec{j} = \sigma\vec{E}$ and conducting means $\vec{j}_\perp = 0$ at ends, so there are Neumann BCs at the boundaries in the x and y directions. So writing $k_{n,m} \equiv \sqrt{n^2\pi^2/a^2 + m^2\pi^2/b^2}$

$$\phi(x,y,z) = Cz + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{n,m}\cos(n\pi x/a)\cos(m\pi y/b)\sinh(k_{n,m}z),$$

and we solve for $A_{n,m}$ and C by using the Fourier transform formulae for $V_0(x,y)$ at $z = b$:

$$A_{n\neq 0, m\neq 0} = \frac{4}{ab\sinh k_{n,m}}\int\int dxdy\cos(n\pi x/a)\cos(n\pi y/b)V(x,y),$$

$$A_{0,m} = \frac{2}{ab\sinh k_{0,m}}L\int dx\int dy\cos(m\pi y/b)V(x,y), \qquad C = \frac{1}{abL}\int\int dxdyV(x,y).$$

• Cylindrical coordinates: $\nabla^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$. Consider first 2d case, setting $z = 0$. $\phi(r,\theta) = R(r)\Theta(\theta)$, then $\nabla^2\Phi/\Phi = (rR)^{-1}\partial_r(r\partial_r R) + (r^2\Theta)^{-1}\partial_\theta^2\Theta$. Solutions of $\nabla^2\phi = 0$ are $\phi = A_0 + B_0\ln r + \sum_{m\neq 0}(A_m r^{|m|} + B_m r^{-|m|})e^{im\theta}$.

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E.g. suppose that there is a cylinder of radius a and the potential at $r = a$ is $V_a(\theta)$. The solution for $r < a$ has $B_m = 0$ and the solution for $r > a$ has $A_m = 0$. The solution on the boundary has e.g. $A_m a^{|m|} = \oint d\theta V_a(\theta) e^{-im\theta}/2\pi$, i.e. the familiar Fourier transform expressions.

For 3d cylindrical, we have $\phi(r, \theta, z) = R(r)\Theta(\theta)Z(z)$ with $Z'' = k^2 Z$ with, taking $u \equiv kr$ the equation for $R(u)$ is the Bessel equation: $R'' + u^{-1}R' + (1 - \nu^2 u^{-2})R = 0$.

E.g. $\phi(r, \theta, z = 0) = 0$, $\phi(r, \theta, z = L) = V_0(r, \theta)$ and $\phi(a, \theta, z) = 0$ has solution $\Theta(\theta) = e^{im\theta}$ and $Z(z) = \sinh(k_{n,m}z)$ and $R_{m,n}(r) = AJ_m(k_{m,n}r) + BN_m(k_{n,m}r)$ where $B = 0$ for the solution be be finite at $r = 0$ and $k_{n,m} = BesseJZero[m, n]/a$.

$$\phi(r, \theta, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{m,n} e^{im\phi} J_m(k_{n,m}r) \sinh(k_{n,m}z).$$

where we get $A_{n,m}$ by inverting the requirement that $\phi(r, \theta, L) = V_0(r, \theta)$. This is done by using orthogonality properties of the Bessel functions:

$$\int_0^a rJ_n(x_{n,m}r/a)J_n(x_{n,m'}r/a)dr = \tfrac{1}{2}a^2 J_{n+1}(x_{n,m})^2 \delta_{m,m'}.$$

• Spherical: $\nabla^2\Psi = r^{-2}\partial_r(r^2\partial_r\Psi) + (r^2\sin\theta)^{-1}\partial_\theta(\sin\theta\partial_\theta\Psi) + (r^2\sin^2\theta)^{-1}\partial_\phi^2\Psi$. Solutions of $\nabla^2\Psi = 0$ are found by taking $\Psi = R(r)\Theta(\theta)\Phi(\phi)$ find $\Phi = e^{im\phi}$ and $\Theta(\theta) = P_\ell^m(\cos\theta)$ the associated Legendre functions. Finally, $R_{\ell,m}(r) = A_{\ell,m}r^{-\ell-1} + B_{\ell,m}r^\ell$. So $\Psi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(A_{\ell,m}r^{-\ell-1} + B_{\ell,m}r^\ell)e^{im\phi}P_\ell^m(\cos\theta)$. Use $Y_{\ell,m} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}$ as the orthonomal, complete basis of functions of (θ, ϕ). Aside you will learn later, in QM, that: $Y_{\ell,m}(\theta, \phi) \sim \langle\theta\phi|\ell m\rangle$ are the eigenstates of L_z and $\vec{L}^2$ in position space basis.

Mathematica: $P_\ell^m(x) = LegendreP[l, m, x]$.

Example: find the potential outside of a sphere of radius a with $V(a, \theta, \phi) = V_0(\Omega)$. Then $\phi = \sum_{\ell,m} A_{\ell,m}r^{-\ell+1}Y_{\ell,m}$ where $A_{\ell,m}a^{-\ell+1} = \int d\Omega Y_{\ell,m}^*(\Omega)V_0(\Omega)$. E.g. for $V_0(\Omega) = V_0 H(\pi/2 - \theta)$ get $A_{\ell,m\neq 0} = 0$, and can do needed integral via Mathematica.

• Wave equation and heat equations in 2d and 3d: $(\partial_t^2 - c^2\nabla^2)\psi = 0$ and $(\partial_t - \chi\nabla^2)\tilde{T} = 0$.

• E.g. heat equation on a rectangle with $S = 0$. Then $\nabla^2 T_{eq}(x, y) = 0$ requires solving the Laplace equation with appropriate BCs. The solution for $\tilde{T}(x, y, t)$ is obtained by separation of variables. For example, with Dirichlet BCs at the ends get

$$\tilde{T} = \sum_{n>0}\sum_{m>0} A_{n,m} e^{-\chi\pi^2(n^2/a^2+m^2/b^2)t} \sin(n\pi x/a)\sin(m\pi y/b)$$

where $A_{n,m}$ is obtained from the initial conditions as

$$A_{n,m} = \frac{4}{ab} \int_0^a dx \int_0^b dy \sin(n\pi x/a) \sin(n\pi y/b)(T_0(x,y) - T_{eq}(x,y)).$$

• Disk drum: $\psi(a,\theta,t) = 0$ with $\psi(r,\theta,0) = z_0(r,\theta)$ and $\partial_t \psi(r,\theta,0) = v_0(r,\theta)$. Separate variables as $\psi = f(t)R(r)e^{im\theta}$ and then $\partial_t^2 f = -\omega_{m,n}^2 f$ where $\omega_{m,n}$ are found from $r^{-1}\partial_r(r\partial_r R) - r^{-2}m^2 R = -\omega_{m,n}^2 R/c^2$, which is Bessel's equation, with $R_{m,n}(r) = AJ_m(\omega_{m,n}r/c) + BY_m(\omega_{m,n}r/c)$, with $B = 0$ to have non-singular behavior at $r = 0$ (or $A = 0$ for non-singular at $r \to \infty$). The n index labels the locations of the zeros of the Bessel's equations solutions, $J_m(j_{m,n}) = 0$.

Traveling wave solutions, e.g. $A_{m,n}e^{i(m\theta - \omega_{m,n}t)} J_m(j_{m,n}r/a)$.

• Oscillations of the surface of a sphere: $(\partial_t^2 - c^2\,\nabla^2)\psi = 0$ in spherical coordinates, taking $r = R$ constant:

$$\psi(t,\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell,m} \cos \omega_\ell t + B_{\ell m} \sin \omega_\ell t) Y_{\ell,m}(\Omega)$$

with $\omega_\ell = c\sqrt{\ell(\ell+1)}/R$.

• Wave equation in spherical coordinates: get

$$\psi = \sum_n \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell m} \cos \omega_{\ell m n} t + B_{\ell m} \sin \omega_{\ell m n} t) R_{\ell n}(r) Y_{\ell,m}(\theta,\phi).$$

where $R'' + 2r^{-1}R' + (k^2 - \ell(\ell+1)r^{-2})R = 0$ is related to the spherical Bessel equation and $k = \omega/c$. The solutions are $R_\ell = j_\ell(kr) + n_\ell(kr)$, where j_ℓ is the solution that works for $r \to 0$. The integer n is determined by some boundary conditions in r, e.g. for Dirichlet boundary conditions it labels the zeros of j_ℓ. E.g. $j_0(x) = \sin x/x$, $j_1(x) = \sin x/x^2 - \cos x/x$, $n_0 = -\cos x/x$, etc.