Physics 105a, Ken Intriligator lecture 17, Nov 30, 2017

• Continue from last time: heat flow equation in $1+1d$: $C(x)\partial_t T = \partial_x(\kappa \partial_x T) + S$, where C is the specific heat, κ is the thermal conductivity, and S is the source of heat energy per unit volume. For κ and C constant, this becomes $\partial_t T = \chi \partial_x^2 T + S/C$, where $\chi = \kappa/C$ is the thermal diffusivity (units of m^2/s e.g. $\chi \sim 10^{-7}$ for water and $\chi \sim 10^{-4}$ for copper. Initial condition in time $= T(x,0) = T_0(x)$. Initial condition in space e.g. D boundary conditions: $T_1(t)$ and $T_2(t)$ at the two ends (or N conditions specifying T_x at either or both ends.

Separation of variables works if there is an equilibrium solution $T_{eq}(x)$, that is time independent. Necessary to have t independent boundary conditions and source. The solution then takes the form $T(x,t) = T_{eq}(x) + \tilde{T}(x,t)$ (where $\tilde{T} \equiv \Delta T$ in the book). Then $\chi T''_{eq}(x) = S/C$ is integrated to find $T_{eq}(x)$ and $\partial_t \tilde{T} = \chi \partial_x^2 \tilde{T}$. As usual, separate variables, taking $\tilde{T} = f(t)\psi(x)$ to get $\partial_t \ln f = \lambda$ and $\chi \partial_x^2 \psi = \lambda \psi$, with λ a constant. Typically the setup gives $\lambda < 0$ to avoid exponentially growing solutions in time and space, so get e.g. (in the DD boundary condition case; for NN get instead $\cos(n\pi x/L)$ solutions).

$$T(x,t) = T_{eq}(x) + \sum_n A_n e^{-\chi t (n\pi/L)^2} \sin(n\pi x/L).$$

The A_n are determined by the initial condition on $T(x, t=0) = T_0(x)$ via $A_n = \frac{2}{L}\int_0^L (T_0(x) - T_{eq}(x)) \sin(n\pi x/L) dx$. Higher n modes are damped out faster; show animation from ch3.nb. The heat flux is $-\kappa \partial_x T$ is non-zero at the ends, which is why the energy inside decreases. If the BCs at both ends are instead Neumann, then we instead have $T(x,t) = \sum_{n=0}^{\infty} A_n e^{-\chi t n^2 \pi^2/L^2} \cos(n\pi x/L)$. There is then no heat flux at the ends.

• Now consider Laplace and Poisson's equations, $\nabla^2 \phi = -\rho$, Or the wave equation in more space dimensions, $\nabla^2 \psi - \frac{1}{c^2}\partial_t^2 \psi = 0$. Or the heat equation in more space dimensions, $\partial_t T = \chi \nabla^2 T$. In all of them we replace the 1d ∂_x^2 with ∇^2. The reason for this is rotational symmetry. Separation of variables is then to take $\psi = f(t)X(x)Y(y)Z(z)$ if the setup is rectangular. Or $\psi = f(t)R(r)\Theta(\theta)\Phi(\phi)$ if the setup is spherical. Or $\psi = f(t)P(\rho)\Phi(\phi)Z(z)$ if it's cylindrical. If it's none of these, there are some other separable coordinate systems. If it's none of those, it might be better to just give it to a computer.

• Solutions of the Laplace equation with N or D boundary conditions are unique: if $\Phi = \phi_1 - \phi_2$ is the difference then $0 = \int_V \Phi \nabla^2 \Phi dV = \int_{\partial V} \Phi \nabla \Phi \cdot d\vec{a} - \int_V \nabla\Phi \cdot \nabla\Phi$, and then the BCs ensure that the $\int_{\partial V}$ term $= 0$, and positivity $\nabla\Phi \cdot \nabla\Phi$ implies that this integral can only vanish if Φ is a constant, which must be zero to satisfy the BCs.

• E.g. 2d rectangular: $\phi(x,y) = X(x)Y(y)$ with $X(x) = C_1 e^{\kappa x} + C_2 e^{-\kappa x}$ and $Y(y) = C_3 e^{i\kappa y} + C_4 e^{-i\kappa y}$, where κ is real or imaginary depending on the BCs. The solutions are sins and coss in one directions, and sinhs and coshs in the other. Example: suppose that BCs are $\phi(x,0) = \phi(y,0) = \phi(x,b) = 0$ and $\phi(a,y) = \phi_A(y)$. Then $Y(0) = Y(b) = 0$ which requires that the oscillatory direction is y, and the exponential direction is x:

$$\phi(x,y) = \sum_n A_n \sinh(n\pi x/b) \sin(n\pi y/b), \qquad A_n \sinh(n\pi a/b) = \frac{2}{b} \int_0^b \phi_A(y) \sin(n\pi y/b) dy.$$