

- Start Chapter 3. Introducing partial differential equations, and their solution by separation of variables. First example, the string wave equation:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)y(x, t) = 0,$$

where $c = \sqrt{\tau/\mu}$ is the wave velocity on the string and $y(x, t)$ is the string's displacement from equilibrium. Aside: This equation comes from $F_y = ma_y$ for the bit of string between x and $x + dx$: write it as $dF_y = dma_y$ where the d is because it is a small element e.g. its mass is $dm = \mu dx$ and $dF_y = F_y(x + dx) - F_y(x) = \partial_x F_y dx$ and $F_y(x) = T \partial y / \partial x$. Aside: the string wave equation can be obtained from the Euler Lagrange equations with

$$S = \int dt \int dx \left(\frac{1}{2} \mu \dot{y}^2 - \frac{1}{2} T y'^2\right).$$

The string wave equation is can be called the wave equation in 1+1 dimensions. There is a similar equation in 2+1 dimensions, describing for example waves on a 2d surface (like a drum head), or in 3+1 dimensions

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi(x, t) = 0$$

where ψ could be the longitudinal displacement of air in a sound wave, or it could be a transverse component of \vec{E} or \vec{B} in a light wave.

- Note that the wave equations are linear, so they can be solved by superposition. One class of solutions are left and right moving traveling waves

$$y(x, t) = f(x + ct) + g(x - ct)$$

which satisfies the wave equation for any functions f and g . For fixed ends (Dirichlet boundary conditions), a traveling wave that hits the end gets reflected back and satisfies $y(x = x_{end}, t) = 0$ by having the reflected wave get flipped upside down. (You probably saw essentially this same thing in a class on waves in the context of the phase shift when light reflects off of a medium with larger index of refraction.)

- Illustrating separation of variables for the string wave equation. Take $y = f(t)\psi(x)$ and plug into the equation. Put all of the t -dependent terms on one side of the equation, and all of the x -dependent terms on the other. The equation has to be satisfied for all (t, x) .

This can only work if both sides are a constant. The upshot is two ordinary differential equations. For the case of the string, both of these equations are simply that of a SHO:

$$\frac{d^2 f}{dt^2} = -\omega^2 f, \quad \frac{d^2 \psi(x)}{dx^2} = -k^2 \psi(x), \quad \omega = ck.$$

Suppose that there are Dirichlet boundary conditions with $\psi(x=0) = \psi(x=L) = 0$. Then the solutions for ψ are labeled by integers (there is no solution unless $k = k_n$ takes special values):

$$\psi_n(x) = C_n \sin(n\pi x/L), \quad \text{i.e.} \quad k_n = 2\pi/\lambda_n = n\pi/L \quad n = 1, 2, 3, \dots$$

where $n = 1$ is the fundamental mode, $n = 2$ is the first excited mode (which has one node in the middle, $n = 3$ is the second excited mode, etc., $\lambda_n = 2L/n$, and C_n are constants. These are the same as the solutions for the wave function of a particle in a 1d box in quantum mechanics, where there $p_n = \hbar k_n$ (because ψ satisfies the same differential equation and the same boundary conditions). Since $k = k_n$, the time-dependent equation has frequency $\omega = \omega_n = ck_n$. We can get a general solution by superposition

$$y(x, t) = \sum_{n=1}^{\infty} (A_n \cos(cn\pi t/L) + B_n \sin(cn\pi t/L)) \sin(n\pi x/L).$$

The A_n and B_n are constants of integration, that can be determined from the initial conditions. If we are told that $y(x, 0) = y_0(x)$ and $\dot{y}(x, 0) = v_0(x)$, with $y_0(x)$ and $v_0(x)$ some given functions, then we can related A_n and B_n to the Fourier coefficients in the Fourier series expansion of these functions:

$$A_n = \frac{2}{L} \int_0^L y_0(x) \sin(n\pi x/L) dx, \quad B_n = \frac{2}{L\omega_n} \int_0^L v_0(x) \sin(n\pi x/L) dx.$$

- Example: take $y_0(x)$ to be a straight line, of slope a for $0 < x < L/2$ and slope $-a$ for $L/2 < x < L$, with $v_0(x) = 0$. So $B_n = 0$ and $A_n = 4aL \sin(n\pi/2)/n^2\pi^2$. Show animation from Dubin cell 3.3 for the first $M = 30$ terms. Looks strange, but it's correct for undamped oscillations.

- Example: Gaussian $y_0(x) = e^{-50(x-L/2)^2/L^2}$, $v_0(x) = 0$. Again, animations from Dubin.

- Dirichlet boundary conditions: $y(0, t) = y_0$, $y(L, t) = y_L$, fixed. Neumann boundary conditions: instead specify the slope at the ends, $y_x(0, t)$ or $y_x(L, t)$. Can have NN, ND,

DN, DD boundary conditions (i.e. either one, at either end), depending on the physical setup.

- Heat flow equation in 1 + 1d: $C(x)\partial_t T = \partial_x(\kappa\partial_x T) + S$, where C is the specific heat, κ is the thermal conductivity, and S is the source of heat energy per unit volume. For κ and C constant, this becomes $\partial_t T = \chi\partial_x^2 T + S/C$, where $\chi = \kappa/C$ is the thermal diffusivity (units of m^2/s e.g. $\chi \sim 10^{-7}$ for water and $\chi \sim 10^{-4}$ for copper. Initial condition in time = $T(x, 0) = T_0(x)$. Initial condition in space e.g. D boundary conditions: $T_1(t)$ and $T_2(t)$ at the two ends (or N conditions specifying T_x at either or both ends).

Separation of variables works if there is an equilibrium solution $T_{eq}(x)$, that is time independent. Necessary to have t independent boundary conditions and source. The solution then takes the form $T(x, t) = T_{eq}(x) + \tilde{T}(x, t)$ (where $\tilde{T} \equiv \Delta T$ in the book). Then $\partial_t \tilde{T} = \chi\partial_x^2 \tilde{T}$. As usual, separate variables, taking $\tilde{T} = f(t)\psi(x)$ to get $\partial_t \ln f = \lambda$ and $\chi\partial_x^2 \psi = \lambda\psi$, with λ a constant. Typically the setup gives $\lambda < 0$ to avoid exponentially growing solutions in time and space, so get e.g. (in the DD boundary condition case; for NN get instead $\cos(n\pi x/L)$ solutions).

$$T(x, t) = T_{eq}(x, t) + \sum_n A_n e^{-\chi t(n\pi/L)^2} \sin(n\pi x/L).$$

- Now consider Laplace and Poisson's equations, $\nabla^2 \phi = -\rho$, Or the wave equation in more space dimensions, $\nabla^2 \psi - \frac{1}{c^2} \partial_t^2 \psi = 0$. Or the heat equation in more space dimensions, $\partial_t T = \chi \nabla^2 T$. In all of them we replace the 1d ∂_x^2 with ∇^2 . The reason for this is rotational symmetry. Separation of variables is then to take $\psi = f(t)X(x)Y(y)Z(z)$ if the setup is rectangular. Or $\psi = f(t)R(r)\Theta(\theta)\Phi(\phi)$ if the setup is spherical. Or $\psi = f(t)P(\rho)\Phi(\phi)Z(z)$ if it's cylindrical. If it's none of these, there are some other separable coordinate systems. If it's none of those, it might be better to just give it to a computer.