Physics 105a, Ken Intriligator lecture 14, Nov 16, 2017

• Damped SHO with general forcing function $f(t)$: $x''+\gamma x'+\omega_0^2 x = f(t)$ has particular solution given by taking the FT of the forcing function then dividing by the differential operator in Fourier space, and then FT-ing back:

$$x_p(t) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}\tilde{f}(\omega)}{-\omega^2 - i\gamma\omega + \omega_0^2}\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} dt' f(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{-\omega^2 - i\gamma\omega + \omega_0^2}.$$

• Write the particular solution above of the damped SHO as

$$x_p(t) = \int_{-\infty}^{\infty} dt' f(t') G(t-t') \qquad G(t-t') \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{-\omega^2 - i\gamma\omega + \omega_0^2}.$$

The quantity $G(t)$ is the Green's function for the SHO differential equation. More generally, Green's function is the particular solution for the case where the source is a delta function. If we take $f(t) = \delta(t)$, we get $x_p(t) = G(t)$. Then, because we can write $f(t) = \int dt' f(t')\delta(t-t')$, we can think of all of the t dependence being in $\delta(t-t')$, with $f(t')$ constants, and get the particular solution for the general case using superposition. This works because the differential equation is linear. The same idea is used in E& M, e.g. we can write $\phi(\vec{x}) = \int d^3\vec{x}'\rho(\vec{x}')/|\vec{x}-\vec{x}'|$ to solve $\nabla^2\phi(\vec{x}) = 4\pi\rho(\vec{x})$, with $G(\vec{x}) = 1/|\vec{x}|$ the Green's function satisfying $\nabla^2 G(\vec{x}) = 4\pi\delta^3(\vec{x})$.

Writing $-\omega^2 - i\gamma\omega + \omega_0^2 \equiv (i\omega + s_1)(i\omega + s_2)$, we can compute $G(t)$ as

$$G(t) = \frac{1}{s_2 - s_1}\int \frac{d\omega}{2\pi} e^{-i\omega t}\left(\frac{1}{i\omega + s_1} - \frac{1}{i\omega + s_2}\right) = \frac{\Theta(t)}{s_1 - s_2}(e^{s_1 t} - e^{s_2 t})$$

$$= \Theta(t)\frac{e^{-\gamma t/2}}{\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}}\sin\left(\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}\, t\right).$$

Note that $\int d\omega$ can be evaluated via Cauchy's theorem, where for $t > 0$ we close the contour with ω in the lower half plane, since then $e^{-i\omega t} \to 0$ for $\omega \to -i\infty$, whereas for $t < 0$ we close in the upper half plane. The poles are at $\omega = -is_1$ and $\omega = -is_2$, where $s_{1,2} = \frac{1}{2}\gamma \pm i\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}$ have positive real part, so the poles are in the lower half plane (since the friction coefficient $\gamma > 0$). This is how we get the $\Theta(t)$.

Likewise, if the differential equation is $L\psi(t) = f(t)$, with $L = \prod_{a=1}^{N}(\frac{d}{dt} - s_a)$ then in Fourier space $L \to \tilde{L}(\omega) = \prod(-i\omega - s_a)$ and

$$G(t) = \int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\tilde{L}(\omega)} = \Theta(t)\sum_{a=1}^{N}\prod_{b\neq a}(s_a - s_b)^{-1} e^{s_a t}$$

where the integral is simply evaluated via Cauchy's theorem, where $\Theta(t)$ comes from the closing of the contour depending on the sign of t, and getting poles only for the $t > 0$ case, and with $\sum_a$ from the sum over the poles at $\omega = is_a$ with residue $(1/2\pi i)\prod_{b\neq a}(s_a - s_b)^{-1}e^{s_a t}$.

• We can solve for the Greens function $g(t, t_0)$ of a differential operator L, i.e. $Lg(t) = \delta(t - t_0)$, by making use of the homogeneous solutions, since for $t \neq t_0$ the equation for $g(t)$ is the homogenous solution. The idea is to write the homogenous solution for both $t < t_0$ and $t > t_0$ in terms of general and different constants of integration $C_{1...N}$ and then relate them so as to get the $\delta(t)$ at $t = t_0$.

Consider e.g. a 2nd order ODE with $L = \frac{d^2}{dt^2} + u_1(t)\frac{d}{dt} + u_0(t)$. For $t > t_0$ and $t < t_0$, g satisfies the homogenous equation, so write e.g. $g(t > t_0) = C_1 x_1(t) + C_2 x_2(t)$. We impose $g(t < t_0) = 0$. Now solve for C_1 and C_2 by making $g(t)$ continuous at $t = t_0$, and matching the first derivative discontinuity to the delta function using $\int_{t_0-\epsilon}^{t_0+\epsilon} dtLg = \int dt\delta(t - t_0) = 1$. The result is

$$g(t, t_0) = \Theta(t - t_0)(x_2(t_0)x_1(t) - x_1(t_0)x_2(t))/W(t_0), \qquad W(t) \equiv x_1'(t)x_2(t) - x_2'(t)x_1(t)$$

with W called the Wronskian. For linearly independent solutions, $W(t) \neq 0$. So

$$x_p(t) = \int_{-\infty}^{t} g(t, t_0)f(t_0)dt_0 = \int_{-\infty}^{t} dt_0 f(t_0)(x_1(t)x_2(t_0) - x_2(t)x_1(t_0))/W(t_0).$$

• Example: $Lx = x'' - nx'/t$. Then $x_1 = 1$ and $x_2 = t^{n+1}/(n+1)$ and $W(t) = x_1'x_2 - x_2'x_1 = -t^n$ and $g(t, t_0) = (n+1)^{-1}(t(t/t_0)^n - t_0)\Theta(t - t_0)$. Taking e.g. $f = t^\alpha\Theta(t)$ then $x_p(t) = \int_0^t g(t, t_0)t_0^\alpha dt_0 = t^{2+\alpha}\Theta(t)/(2+\alpha)(1+\alpha-n)$.

• Generalize this method of patching together the homogenous solutions to solve $Lg(t, t_0) = \delta(t - t_0)$ to the case where L is an N-th order differential operator. There are N coefficients C_i and we get N equations by imposing continuity of g and its first $N - 2$ derivatives at $t = t_0$, while $d^{N-1}g/dt^{N-1}$ has a discontinuity to give the $\delta(t)$. Since there are N equations for N unknowns C_N, they can be solved.

• The Greens function gives a way to invert L: taking $Lx_p = f$, we invert via $x_p = L^{-1}f = \int g(t, t_0)f(t_0)dt_0$. The homogeneous solution is in the nullspace of L. We eliminate that by taking $x(t) = 0$ for t before when the forcing turns on. By taking t to be a grid, L becomes a matrix and the $\int dt_0 g(t, t_0)$ is seen to be matrix multiplication by L^{-1}. Greens functions as inversion of the matrix L can be implemented via mathematica as a way to numerically solve for the particular solutions.

• Greens functions for boundary value problems, e.g. $\phi''(x) = \rho(x)$, with some specified values of ϕ at the ends of the region, $\phi(a) = V_a$, $\phi(b) = V_b$. The Greens function satisfies $Lg(x, x_0) = \delta(x - x_0)$ and $g(a, x_0) = g(b, x_0) = 0$. This can be solved similarly to the initial value case: e.g. we use the homogenous solution for $x < x_0$ and $x > x_0$, and match the coefficients in the two regions to give continuity up to the effect of the delta function. Use $\bar{\phi}_1 = \phi_1(x) - \phi_1(a)\phi_2(x)/\phi_2(a)$ and $\bar{\phi}_2 = \phi_1(x) - \phi_1(b)\phi_2(x)/\phi_2(b)$ to satisfy $\phi_1(a) = \phi_2(b) = 0$. Then $g(x < x_0) = C_1\bar{\phi}_1(x)$, and $g(x > x_0) = C_2\bar{\phi}_2(x)$. Imposing continuity of g at x_0, and $g'(x > x_0)|_{x\to x_0} - g'(x < x_0)|_{x\to x_0} = 1$ gives $g(x < x_0) = -\bar{\phi}_2(x_0)\phi_1(x)/W(x_0)$ and $g(x > x_0) = -\bar{\phi}_1(x_0)\bar{\phi}_2(x)/W(x_0)$ with $W(x) = \bar{\phi}_1'(x)\bar{\phi}_2 - \bar{\phi}_2'\bar{\phi}_1$.

• In the last lecture, we discussed Greens functions for initial value problems $Lx(t) = f(t)$, or boundary value problems $L\phi(x) = \rho(x)$. The Greens functions satisfy $Lg(t, t_0) = \delta(t - t_0)$, or $Lg(x, x_0) = \delta(x - x_0)$, respectively. For initial value problems, we specify the initial conditions, e.g. that $x = 0$ before the forcing function turns on, whereas for the boundary value problems we specify that $g = 0$ at the two boundaries (this is for Dirichlet boundary conditions; for Neumann boundary conditions we instead specify that $g' = 0$ at a boundary).

As mentioned in the last lecture, writing $x_p = \int g(t, t_0)f(t_0)dt_0$ or $\phi = \int g(x, x_0)\rho(x_0)dx_0$ can be understood as inverting the ODE by multiplying both sides by L^{-1}. By taking time or space to be a lattice grid, L becomes literally a matrix and the Greens function is literally what one gets by inverting that matrix to solve e.g. $L\phi = \rho$ via $\phi = L^{-1}\rho$.

• Consider $L\phi = \phi'' + u_1\phi' + u_0\phi = \rho$ and take space $a < x < b$ to be a grid $x \to x_n = a + n\Delta x$, with $\Delta x = (b - a)/M$. When we take space or time to be a grid, we have some options for how to write derivatives, which all reduce to the ordinary derivative in the limit $M \to \infty$. Apparent differences between those options can be called lattice artifacts, but for finite M some are more convenient than others, i.e. some artifacts go away faster with $1/M$ than others. Write $\phi(x_n)$ or $\phi(t_n)$ as ϕ_n. As discussed in Table 2.3 of Dubin, there are three options for first derivatives, forward difference, backward difference, centered difference, with centered difference most accurate: $\phi_n' \approx (\phi_{n+1} - \phi_{n-1})/2\Delta x$, and $\phi_n'' \approx (\phi_{n+1} - 2\phi_n + \phi_{n-1})/\Delta x^2$. Using centered difference derivatives, get

$$L_{n,m} = (\delta_{n+1,m} - 2\delta_{n,m} + \delta_{n-1,m})\Delta x^{-2} + \tfrac{1}{2}u_1(n)(\delta_{n+1,m} - \delta_{n-1,m})\Delta x^{-1} + \delta_{n,m}u_0(n).$$

Illustrate this in Mathematica and $\phi = L^{-1}\rho$. Compared with the shooting method, this has some advantages - but it only works for linear differential equations.