

- Example: $f(t) = e^{-t}\Theta(t) \leftrightarrow \tilde{f}(\omega) = (1 - i\omega)^{-1}$.
- Periodic version of the delta function: $\delta^{P,T}(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT) = \sum_{n=-\infty}^{\infty} e^{-i2\pi nt/T}$.

More generally, if a function is periodic, $f(t + T) = f(t)$ then the Fourier integral and Fourier sum expressions are related via

$$f(t) = \int \tilde{f}(\omega) e^{-i\omega t} \frac{d\omega}{2\pi} = \sum_n \tilde{f}_n e^{-2\pi i n t/T} \quad \text{with} \quad \tilde{f}(\omega) = \sum_n 2\pi \tilde{f}_n \delta(\omega - 2\pi n/T).$$

- Damped SHO with general forcing function $f(t)$: $x'' + \gamma x' + \omega_0^2 x = f(t)$ has particular solution given by taking the FT of the forcing function then dividing by the differential operator in Fourier space, and then FT-ing back:

$$x_p(t) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t} \tilde{f}(\omega)}{-\omega^2 - i\gamma\omega + \omega_0^2} \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} dt' f(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-\omega^2 - i\gamma\omega + \omega_0^2}.$$

- Write the particular solution above of the damped SHO as

$$x_p(t) = \int_{-\infty}^{\infty} dt' f(t') G(t - t') \quad G(t - t') \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-\omega^2 - i\gamma\omega + \omega_0^2}.$$

The quantity $G(t)$ is the Green's function for the SHO differential equation. More generally, Green's function is the particular solution for the case where the source is a delta function. If we take $f(t) = \delta(t)$, we get $x_p(t) = G(t)$. Then, because we can write $f(t) = \int dt' f(t') \delta(t - t')$, we can think of all of the t dependence being in $\delta(t - t')$, with $f(t')$ constants, and get the particular solution for the general case using superposition. This works because the differential equation is linear. The same idea is used in E& M, e.g. we can write $\phi(\vec{x}) = \int d^3 \vec{x}' \rho(\vec{x}') / |\vec{x} - \vec{x}'|$ to solve $\nabla^2 \phi(\vec{x}) = 4\pi \rho(\vec{x})$, with $G(\vec{x}) = 1/|\vec{x}|$ the Green's function satisfying $\nabla^2 G(\vec{x}) = 4\pi \delta^3(\vec{x})$.

Writing $-\omega^2 - i\gamma\omega + \omega_0^2 \equiv (i\omega + s_1)(i\omega + s_2)$, we can compute $G(t)$ as

$$\begin{aligned} G(t) &= \frac{1}{s_2 - s_1} \int \frac{d\omega}{2\pi} e^{-i\omega t} \left(\frac{1}{i\omega + s_1} - \frac{1}{i\omega + s_2} \right) = \frac{\Theta(t)}{s_1 - s_2} (e^{s_1 t} - e^{s_2 t}) \\ &= \Theta(t) \frac{e^{-\gamma t/2}}{\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}} \sin \left(\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2} t \right). \end{aligned}$$

Note that $\int d\omega$ can be evaluated via Cauchy's theorem, where for $t > 0$ we close the contour with ω in the lower half plane, since then $e^{-i\omega t} \rightarrow 0$ for $\omega \rightarrow -i\infty$, whereas for

$t < 0$ we close in the upper half plane. The poles are at $\omega = -is_1$ and $\omega = -is_2$, where $s_{1,2} = \frac{1}{2}\gamma \pm i\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}$ have positive real part, so the poles are in the lower half plane (since the friction coefficient $\gamma > 0$). This is how we get the $\Theta(t)$.

Likewise, if the differential equation is $L\psi(t) = f(t)$, with $L = \prod_{a=1}^N (\frac{d}{dt} - s_a)$ then in Fourier space $L \rightarrow \tilde{L}(\omega) = \prod(-i\omega - s_a)$ and

$$G(t) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\tilde{L}(\omega)} = \Theta(t) \sum_{a=1}^N \prod_{b \neq a} (s_a - s_b)^{-1} e^{s_a t}$$

where the integral is simply evaluated via Cauchy's theorem, where $\Theta(t)$ comes from the closing of the contour depending on the sign of t , and getting poles only for the $t > 0$ case, and with \sum_a from the sum over the poles at $\omega = is_a$ with residue $(1/2\pi i) \prod_{b \neq a} (s_a - s_b)^{-1} e^{s_a t}$.

- We can solve for the Greens function $g(t, t_0)$ of a differential operator L , i.e. $Lg(t) = \delta(t - t_0)$, by making use of the homogeneous solutions, since for $t \neq t_0$ the equation for $g(t)$ is the homogenous solution. The idea is to write the homogenous solution for both $t < t_0$ and $t > t_0$ in terms of general and different constants of integration $C_{1...N}$ and then relate them so as to get the $\delta(t)$ at $t = t_0$.

Consider e.g. a 2nd order ODE with $L = \frac{d^2}{dt^2} + u_1(t)\frac{d}{dt} + u_0(t)$. For $t > t_0$ and $t < t_0$, g satisfies the homogenous equation, so write e.g. $g(t > t_0) = C_1 x_1(t) + C_2 x_2(t)$. We impose $g(t < t_0) = 0$. Now solve for C_1 and C_2 by making $g(t)$ continuous at $t = t_0$, and matching the first derivative discontinuity to the delta function using $\int_{t_0-\epsilon}^{t_0+\epsilon} dt Lg = \int dt \delta(t - t_0) = 1$. The result is

$$g(t, t_0) = \Theta(t - t_0)(x_2(t_0)x_1(t) - x_1(t_0)x_2(t))/W(t_0), \quad W(t) \equiv x_1'(t)x_2(t) - x_2'(t)x_1(t)$$

with W called the Wronskian. For linearly independent solutions, $W(t) \neq 0$. So

$$x_p(t) = \int_{-\infty}^t g(t, t_0) f(t_0) dt_0 = \int_{-\infty}^t dt_0 f(t_0)(x_1(t)x_2(t_0) - x_2(t)x_1(t_0))/W(t_0).$$

- Example: $Lx = x'' - nx'/t$. Then $x_1 = 1$ and $x_2 = t^{n+1}/(n+1)$ and $W(t) = x_1'x_2 - x_2'x_1 = -t^n$ and $g(t, t_0) = (n+1)^{-1}(t(t/t_0)^n - t_0)\Theta(t - t_0)$. Taking e.g. $f = t^\alpha\Theta(t)$ then $x_p(t) = \int_0^t g(t, t_0)t_0^\alpha dt_0 = t^{2+\alpha}\Theta(t)/(2+\alpha)(1+\alpha-n)$.

- Generalize this method of patching together the homogenous solutions to solve $Lg(t, t_0) = \delta(t - t_0)$ to the case where L is an N-th order differential operator. There are N coefficients C_i and we get N equations by imposing continuity of g and its first $N - 2$

derivatives at $t = t_0$, while $d^{N-1}g/dt^{N-1}$ has a discontinuity to give the $\delta(t)$. Since there are N equations for N unknowns C_N , they can be solved.

- The Greens function gives a way to invert L : taking $Lx_p = f$, we invert via $x_p = L^{-1}f = \int g(t, t_0)f(t_0)dt_0$. The homogeneous solution is in the nullspace of L . We eliminate that by taking $x(t) = 0$ for t before when the forcing turns on. By taking t to be a grid, L becomes a matrix and the $\int dt_0g(t, t_0)$ is seen to be matrix multiplication by L^{-1} . Greens functions as inversion of the matrix L can be implemented via mathematica as a way to numerically solve for the particular solutions.

- Greens functions for boundary value problems, e.g. $\phi''(x) = \rho(x)$, with some specified values of ϕ at the ends of the region, $\phi(a) = V_a$, $\phi(b) = V_b$. The Greens function satisfies $Lg(x, x_0) = \delta(x - x_0)$ and $g(a, x_0) = g(b, x_0) = 0$. This can be solved similarly to the initial value case: e.g. we use the homogenous solution for $x < x_0$ and $x > x_0$, and match the coefficients in the two regions to give continuity up to the effect of the delta function. Use $\bar{\phi}_1 = \phi_1(x) - \phi_1(a)\phi_2(x)/\phi_2(a)$ and $\bar{\phi}_2 = \phi_1(x) - \phi_1(b)\phi_2(x)/\phi_2(b)$ to satisfy $\phi_1(a) = \phi_2(b) = 0$. Then $g(x < x_0) = C_1\bar{\phi}_1(x)$, and $g(x > x_0) = C_2\bar{\phi}_2(x)$. Imposing continuity of g at x_0 , and $g'(x > x_0)|_{x \rightarrow x_0} - g'(x < x_0)|_{x \rightarrow x_0} = 1$ gives $g(x < x_0) = -\bar{\phi}_2(x_0)\phi_1(x)/W(x_0)$ and $g(x > x_0) = -\bar{\phi}_1(x_0)\bar{\phi}_2(x)/W(x_0)$ with $W(x) = \bar{\phi}_1'(x)\bar{\phi}_2 - \bar{\phi}_2'\bar{\phi}_1$.