Physics 105a, Ken Intriligator lecture 12, Nov 9, 2017

- Recall from last time: Fourier transforms:

$$f(t)=\int_{-\infty}^{\infty}\tilde f(\omega)e^{-i\omega t}\frac{d\omega}{2\pi}\qquad\leftrightarrow\qquad \tilde f(\omega)=\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt.$$

Also, there are similar formulae for Fourier transforms in space, with a conventional minus sign difference (so combining gives traveling waves moving to the right):

$$f(x)=\int_{-\infty}^{\infty}\tilde f(k)e^{ikx}\frac{dk}{2\pi}\qquad\leftrightarrow\qquad \tilde f(k)=\int_{-\infty}^{\infty}f(x)e^{-ikx}dx.$$

We discussed $f(t)\leftrightarrow\tilde f(\omega)$ examples

$$\delta(t)\leftrightarrow 1$$

$$1\leftrightarrow 2\pi\delta(\omega)$$

$$even\leftrightarrow even$$

$$odd\leftrightarrow odd$$

$$\frac{d}{dt}\leftrightarrow -i\omega$$

$$t/|t|\leftrightarrow 2i/\omega$$

$$\Theta(t)=\tfrac{1}{2}(1+t/|t|)\leftrightarrow \pi\delta(\omega)+(i/\omega)$$

Today we will also discuss

$$t\leftrightarrow i\frac{d}{d\omega}$$

$$convolution\leftrightarrow multiplication$$

$$thin\leftrightarrow fat$$

Fourier transforms convert $\frac{d}{dt}\to -i\omega$ and $\int dt\to 1/(-i\omega)$ up to constants. Also, they convert convolutions to multiplication: if $h(t)=\int dt_1 f(t_1)g(t-t_1)$, then $\tilde h(\omega)=\tilde f(\omega)\tilde g(\omega)$.

Recall $f(t)=\delta(t)\leftrightarrow\tilde f(\omega)=1$ and $f(t)=1\leftrightarrow\tilde f(\omega)=2\pi\delta(\omega)$. Now consider the FT of $H(t)=\Theta(t)$. Since the FT converts $\frac{d}{dt}\to -i\omega$, and $\frac{d}{dt}\Theta(t)=\delta(t)$, we might guess that the FT of $H(t)$ is i/ω. As we discussed last time, and we can also see from $\tilde H(\omega)=\int_0^\infty dt(\cos\omega t+i\sin\omega t)=\pi\delta(\omega)+i\omega^{-1}$. Again, $\frac{d}{dt}\to -i\omega$ acts on this to give 1, as expected, since $\omega\delta(\omega)=0$. We discuss this last time in terms of the FT of an even or odd function being also even or odd: if $f(-t) = \pm f(t)$ then $f(-\omega) = \pm f(\omega)$. Then $H(t) = \frac{1}{2}(1 + \text{sign}(t))$, and the FTs are $\frac{1}{2}1 \to \pi\delta(\omega)$ and $\frac{1}{2}\text{sign}(t) \to i/\omega$.

- Units of $f(t)$ vs $\tilde{f}(\omega)$.

- Parseval's result, and interpretation as inner product of function with itself in either basis. Another way to say it is that the Fourier transform is unitary (recall a matrix or operator U is unitary if $UU^\dagger = \mathbf{1}$; the eigenvalues of such an operator are $e^{i\phi}$ with ϕ real). In QM this implies that if the wave function is properly normalized in position space, it'll automatically be properly normalized in momentum i.e. wavenumber space. Comment about this in the various examples.

- Recall that $f(t) = \delta(t) \leftrightarrow \tilde{f}(\omega) = 1$, and thus $\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}$. Taking the integral instead from $-a$ to a, where a is real and positive, gives $\delta(t) = \lim_{a\to\infty} \sin(at)/\pi t$. Verify for all a that the area under this curve is 1, and that its shape verifies $\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}$. Now, as a next example, let's consider the FT of $f(t) = \sin(at)/\pi t$, with a real and positive. The FT gives $f(t) \to \tilde{f}(\omega) = \Theta(a - |\omega|)$; show how to do the integral by using Cauchy's theorem (similar to a midterm question), deforming the pole to be above the contour. For $a \to \infty$ this indeed becomes $FT(\delta(t)) \to 1$. Note that, varying a this is an example of thin$\leftrightarrow$ fat.

- (Show Mathematica file and see comments there about units, the derivative of the delta function, etc.)

- Next example: $f(t) = 1/(1 + s^2t^2)$ gives (either via Mathematica or Cauchy's)

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} (1 + s^2t^2)^{-1} e^{i\omega t} dt = \pi e^{-|\omega s|}/|s|.$$

Recall how to do it via Cauchy's theorem: we can think if the integral as in the complex t plane, and can close the contour for $t \to +\infty$ if $\omega > 0$, or $t \to -i\infty$ if $\omega < 0$. The poles of the integrand are at $t_\pm = \pm i/|s|$, and we can write $f(t) = 1/s^2(t - t_+)(t - t_-)$, so the residue at $t_\pm$ is $\pm 1/s^2(t_+ - t_-) = 1/2|s|i$. For $\omega > 0$ we get the pole at t_+ and for $\omega < 0$ we get the pole at t_-, so $\tilde{f}(\omega) = 2\pi i(1/2|s|i)e^{-|\omega s|}$.

- Example: $f(t) = e^{-t}\Theta(t) \leftrightarrow \tilde{f}(\omega) = (1 - i\omega)^{-1}$.

- Example: $f(x) = e^{i\bar{k}x}G(x - \bar{x}, \sigma)$, where $G(x - \bar{x}, \sigma) = (2\pi\sigma^2)^{-1/2}\exp(-(x - \bar{x})^2/2\sigma^2)$ is the Gaussian normal distribution with mean $\bar{x}$ and standard deviation σ has Fourier transform given by another Gaussian: $\tilde{f}(k) = e^{-ik\bar{x}}G(k - \bar{k}, \tilde{\sigma})$ with $\tilde{\sigma}\sigma = 1$. Narrow in position space means broad in wavenumber space, and vice-versa. This fits with what we saw in Fourier series: the more edges or sharp functions require larger coefficients for the higher frequency modes. In QM, where $p = \hbar k$, this becomes the uncertainty principle: $\Delta p \Delta x \geq \hbar/2$, where the inequality is saturated for Gaussians. The factor of two arises from how we calculate $\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$, and likewise for Δp.

Moral of the story: more localized in space means broader in Fourier coefficient space, and visa-versa.