Physics 105a, Ken Intriligator lecture 11, Nov 7, 2017

• Recall from last time: Fourier transforms:

$$f(t)=\int_{-\infty}^{\infty}\tilde f(\omega)e^{-i\omega t}\frac{d\omega}{2\pi}\qquad\leftrightarrow\qquad \tilde f(\omega)=\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt.$$

Also, there are similar formulae for Fourier transforms in space, with a conventional minus sign difference (so combining gives traveling waves moving to the right):

$$f(x)=\int_{-\infty}^{\infty}\tilde f(k)e^{ikx}\frac{dk}{2\pi}\qquad\leftrightarrow\qquad \tilde f(k)=\int_{-\infty}^{\infty}f(x)e^{-ikx}dx.$$

Some people, and Mathematica, use $\tilde f(\omega)_{there}=\sqrt{2\pi}\tilde f(\omega)_{here}$, which has the advantage that the formulas look more symmetric: both integrals then have a $1/\sqrt{2\pi}$. Personally, I prefer the above normalization. Physically, it makes sense that the $d\omega$ always comes with a $1/2\pi$: it arises from counting modes in a periodic box. For example, consider $f(t)=e^{-i\omega t}$ and require it to have periodicity $t\to t+T$, then $\omega=\omega_n=2\pi n/T$, and nearby modes have $d\omega=2\pi dn/T$, so $\sum_n\to\int dn=T(d\omega/2\pi)$. Likewise, in space, we can consider $e^{i(\vec k\cdot\vec x-\omega t)}$ and periodicity in space gives $k_i=2\pi n_i/L_i$, so $\sum_{n_i}\to\int L_i dk_i/2\pi$. So in time and space we have $TV\int d\omega d^3\vec k/(2\pi)^4$. Aside: in relativity frequency and wave-number combine into a 4-vector $k^\mu=(\omega,\vec k)$ where we set $c=1$ and likewise $x^\mu=(t,\vec k)$, and then we see that $\omega t-\vec k\cdot\vec x$ is Lorentz invariant. The integration over spacetime or frequency space, $dtd^3\vec x=d^4x$ and $d\omega d^3\vec k=d^4k$, are also Lorentz invariant (observers moving at relative velocities get different $d\omega$ and $d^3\vec k$, but the product is invariant).

Let's show that the above makes sense by plugging $\tilde f(\omega)$ back into $f(t)$:

$$f(t)=\int_{-\infty}^{\infty}e^{-i\omega t}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dt' f(t')e^{i\omega t'}f(t')\equiv\int K(t'-t)f(t')dt',\qquad K(t')=\int\frac{d\omega}{2\pi}e^{i\omega t'}.$$

This works because (with " = " because we could, but won't, add various legal disclaimers)

$$\int\frac{d\omega}{2\pi}e^{i\omega t}\qquad \text{“}=\text{”}\qquad \delta(t),\qquad \text{i.e.}\quad f(t)=\delta(t)\leftrightarrow\tilde f(\omega)=1$$

• Note that the Fourier transform is a linear operation: $FT[f(t)]=\tilde f(\omega)$ satisfies $FT[\sum_i c_i f_i]=\sum_i c+iFT[f_i]$. So is the inverse transformation, $(FT)^{-1}\tilde f=f$, and $(FT)(FT)^{-1}=1$ for sufficiently nice functions. There is a nice way to represent all of this in QM notation:

$$\psi(x)=\langle x|\psi\rangle,\qquad \tilde\psi(k)=\langle k|\psi\rangle,$$

are like representing the same vector (the quantity ψ) in terms of different basis vectors (either $|x\rangle$ for position space or $|kl\rangle$ for wavenumber = momentum space = Fourier transform space). The different basis vectors are related by $\langle x|k\rangle = \langle k|x\rangle^* = e^{ikx}$ and $\langle x'|x\rangle = \delta(x-x')$ and $\langle k'|k\rangle = 2\pi\delta(k'-k)$. The fact that the basis is complete is written as

$$\mathbf{1} = \int \frac{dk}{2\pi}|k\rangle\langle k| = \int dx|x\rangle\langle x|.$$

• Fourier transforms convert $\frac{d}{dt} \to -i\omega$ and $\int dt \to 1/(-i\omega)$ up to constants. Also, they convert convolutions to multiplication: if $h(t) = \int dt_1 f(t_1)g(t-t_1)$, then $\tilde{h}(\omega) = \tilde{f}(\omega)\tilde{g}(\omega)$.

• Recall $f(t) = \delta(t) \leftrightarrow \tilde{f}(\omega) = 1$ and $f(t) = 1 \leftrightarrow \tilde{f}(\omega) = 2\pi\delta(\omega)$. Now consider the FT of $H(t) = \Theta(t)$. Since the FT converts $\frac{d}{dt} \to -i\omega$, and $\frac{d}{dt}\Theta(t) = \delta(t)$, we might guess that the FT of $H(t)$ is i/ω. Actually, $\tilde{H}(\omega) = \int_0^\infty dt(\cos\omega t + i\sin\omega t) = \pi\delta(\omega) + i\omega^{-1}$. Note that $-i\omega$ acts on this to give 1, as expected, since $\omega\delta(\omega) = 0$.

• Note that the FT of an even or odd function is also even or odd: if $f(-t) = \pm f(t)$ then $f(-\omega) = \pm f(\omega)$. For exa $H(t) = \frac{1}{2}(1 + \mathrm{sign}(t))$ with 1 even and sign odd. In the FT, $\frac{1}{2}1 \to \pi\delta(\omega)$ and $\frac{1}{2}\mathrm{sign}(t) \to i/\omega$.