Physics 105a, Ken Intriligator lecture 10, October 31, 2017

• Recall periodic $f(t+T)=f(t)$ has a Fourier expansion

$$f(t)=\sum_{n=-\infty}^{\infty}\tilde{f}_n e^{-2\pi int/T}=a_0+\sum_{n=1}^{\infty}(a_n\cos(2\pi nt/T)+b_n\sin(2\pi nt/T)).$$

With

$$\tilde{f}_n=\int_{t_0}^{T+t_0}\frac{dt}{T}f(t)e^{2\pi int/T}.$$

Or equivalently

$$a_0=\int_{t_0}^{T+t_0}\frac{dt}{T}f(t),\quad a_{m>0}=2\int_{t_0}^{T+t_0}\frac{dt}{T}f(t)\cos(2\pi mt/T),\quad b_{m>0}=2\int_{t_0}^{T+t_0}\frac{dt}{T}f(t)\sin(2\pi mt/T),$$

Note that, for real $f(t)$, $\tilde{f}_n=\tilde{f}^*_{-n}$. The exponential and sin and cos forms are related by $\tilde{f}_0=a_0$, $\tilde{f}_{n>0}=\frac{1}{2}(a_n+ib_n)$, $\tilde{f}_{n<0}=\frac{1}{2}(a_{-n}-ib_{-n})$.

• Solving boundary value problems using Fourier series, e.g. for the shaken spring demo. Examples with $\phi(x)$ and $\partial_x^2\phi=-\rho(x)$ with Dirichlet (fixed value) and boundary, say $\phi(0)=\phi_0$ and $\phi(L)=\phi_1$, do expansion in $\sin(n\pi x/L)$. For Neumann boundary conditions the derivatives at the endpoints are instead specified, and then we can instead expand in $\cos(n\pi x/L)$.

Example: take $\rho(x)=Ax(L-x)^2$ and solve it for $x\in[0,L]$. Let's take the odd periodic extension, since $\rho(0)=\rho(L)=0$, so $\rho(x)=\sum_n\tilde{\rho}_n\sin(n\pi x/L)$ and use Mathematica to show $\tilde{\rho}_n=4AL^3(2+(-1)^n)/n^3\pi^3$. So the particular solution has $\phi_p(x)=\sum_n\tilde{\phi}_n\sin(n\pi x/L)$ with $\tilde{\phi}_n=\tilde{\rho}_n/(n\pi/L)^2=4AL^5(2+(-1)^n)/n^5$. Very nicely convergent as $1/n^5$. Check units: expect $\rho\sim Q/L^3$ and $\phi\sim Q/L$ so $A\sim Q/L^6$; works. The electric field contribution is then $\vec{E}_p(x)=-\hat{x}\partial_x\phi_p=-\hat{x}\sum_n(n\pi/L)\tilde{\phi}_n\cos(n\pi x/L)$.

• The step function is called HeavisideTheta[t] $\equiv H[t]\equiv\Theta[t]$; it is 1 for $t>0$ and 0 for $t<0$. Note that $\frac{d}{dt}H(t)=\delta(t)$: this is called the Dirac delta function, and it is zero everywhere, and infinite at $t=0$, such that $\int_a^b dt\delta(t)=H(b)-H(a)$, i.e. it is 1 if the interval $[a,b]$ contains the origin, and 0 otherwise. In words: the delta function is zero everywhere except where its argument is zero, and it blows up there such that the area is 1. It can be defined from various smooth functions via a limiting procedure, e.g. from the Gaussian normal distribution with standard deviation $\sigma\to 0$.

You might have already met the delta function in your E&M class, in the context of point charges q_i at positions $\vec{x}_i$ having density $\rho=\sum_i q_i\delta^3(\vec{x}-\vec{x}_i)$, where $\delta^3(\vec{x})\equiv$

$\delta(x)\delta(y)\delta(z)$. Taking $\phi(\vec{x}) = q/r$ gives $\vec{E} = -\nabla\phi = q\hat{r}/r^2$ and Gauss' law gives $\int_V dV\,\nabla\cdot\vec{E} = \oint_{\partial V}\vec{E}\cdot d\vec{a} = 4\pi q$ provided that V includes the origin. We see that $\nabla^2(1/r) = -4\pi\delta^3(\vec{r})$. A charge distribution of point particles has $\phi(\vec{x}) = \sum_i q_i/|\vec{x}-\vec{x}_i|$ which solves Poisson's equation $\nabla^2\vec{\phi} = -4\pi\vec{\rho}$ with $\rho = \sum_i q_i\delta^3(\vec{x}-\vec{x}_i)$.

• Compare $\delta(x-y)$ and $\delta_{n,m}$. For any function $f(x)$, note that $\int dy f(y)\delta(x-y) = f(x)$. Likewise, for any F_m, $\sum_n F_m\delta_{n,m} = F_n$.

• Fourier transforms:

$$f(t) = \int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} \qquad \leftrightarrow \qquad \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt.$$

Some people, and Mathematica, use $\tilde{f}(\omega)_{there} = \sqrt{2\pi}\tilde{f}(\omega)_{here}$, which has the advantage that the formulas look more symmetric: both integrals then have a $1/\sqrt{2\pi}$. Personally, I prefer the above normalization. Physically, it makes sense that the $d\omega$ always comes with a $1/2\pi$: it arises from counting modes in a periodic box. For example, consider $f(t) = e^{-i\omega t}$ and require it to have periodicity $t\to t+T$, then $\omega = \omega_n = 2\pi n/T$, and nearby modes have $d\omega = 2\pi dn/T$, so $\sum_n \to \int dn = T(d\omega/2\pi)$.

Also, there are similar formulae for Fourier transforms in space, with a conventional minus sign difference (so combining gives traveling waves moving to the right):

$$f(x) = \int_{-\infty}^{\infty}\tilde{f}(k)e^{ikx}\frac{dk}{2\pi} \qquad \leftrightarrow \qquad \tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx.$$

Let's show that the above makes sense by plugging $\tilde{f}(\omega)$ back into $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty} dt' f(t')e^{i\omega t'}f(t') \equiv \int K(t'-t)f(t')dt', \qquad K(t') = \int\frac{d\omega}{2\pi}e^{i\omega t'}.$$

This works because (with " = " because we could, but won't, add various legal disclaimers)

$$\int\frac{d\omega}{2\pi}e^{i\omega t} \qquad \text{“} = \text{”} \qquad \delta(t), \qquad \text{i.e.} \quad f(t) = \delta(t) \leftrightarrow \tilde{f}(\omega) = 1$$

Note also that the Fourier transform has $\frac{d}{dt}f(t) \to -i\omega\tilde{f}(\omega)$ so one seemingly obtains $H(t)\to i/\omega$. Actually this requires some care and the answer is $H(t)\to i/\omega + \pi\delta(\omega)$.

ended here

• Example: $f(t) = 1/(1+s^2t^2)$ gives (either via Mathematica or Cauchy's theorem)

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty}(1+s^2t^2)^{-1}e^{i\omega t}dt = \pi e^{-|\omega s|}/|s|.$$

Recall how to do it via Cauchy's theorem: we can think if the integral as in the complex t plane, and can close the contour for $t \to +\infty$ if $\omega > 0$, or $t \to -i\infty$ if $\omega < 0$. The poles of the integrand are at $t_\pm = \pm i/|s|$, and we can write $f(t) = 1/s^2(t-t_+)(t-t_-)$, so the residue at $t_\pm$ is $\pm 1/s^2(t_+ - t_-) = 1/2|s|i$. For $\omega > 0$ we get the pole at t_+ and for $\omega < 0$ we get the pole at t_-, so $\tilde{f}(\omega) = 2\pi i(1/2|s|i)e^{-|\omega s|}$.

- Example: $f(t) = e^{-t}\Theta(t) \leftrightarrow \tilde{f}(\omega) = (1-i\omega)^{-1}$.

- Fourier transforms convert $\frac{d}{dt} \to -i\omega$ and $\int dt \to 1/(-i\omega)$. Also, they convert convolutions to multiplication: if $h(t) = \int dt_1 f(t_1)g(t-t_1)$, then $\tilde{h}(\omega) = \tilde{f}(\omega)\tilde{g}(\omega)$.

- Periodic version of the delta function: $\delta^{P,T}(t) = \sum_{m=-\infty}^{\infty} \delta(t-mT) = \sum_{n=-\infty}^{\infty} e^{-i2\pi nt/T}$.

More generally, if a function is periodic, $f(t+T) = f(t)$ then the Fourier integral and Fourier sum expressions are related via

$$f(t) = \int \tilde{f}(\omega)e^{-i\omega t}\frac{d\omega}{2\pi} = \sum_n \tilde{f}_n e^{-2\pi i n t/T} \qquad \text{with} \quad \tilde{f}(\omega) = \sum_n 2\pi \tilde{f}_n \delta(\omega - 2\pi n/T).$$

- Damped SHO with general forcing function $f(t)$: $x'' + \gamma x' + \omega_0^2 x = f(t)$ has particular solution given by

$$x_p(t) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}\tilde{f}(\omega)}{-\omega^2 - i\gamma\omega + \omega_0^2}\frac{d\omega}{2\pi}.$$