105a Homework 3, due before Oct 22, 2017 at 8pm

1. This exercise will be about the function $\Gamma[z]$ (called Gamma[z] in mathematica).

   a) For $Re[z] > 0$, $\Gamma[z]$ can be defined by an integral: $\Gamma[z] = \int_0^\infty dt t^{z-1} e^{-t}$ (it appears inKen.Lec4.nb, with the integral named "Grandma"). Use integration by parts to show that $\Gamma[z+1] = z\Gamma[z]$. Also verify by doing the integral that $\Gamma[1] = 1$. It follows that $\Gamma[n+1] = n!$ for integer n. You can write this up in your mathematica notebook, but you should not need to use mathematica to do any calculations.

   b) Verify by a substitution that $\Gamma[1/2] = \int_{-\infty}^{\infty} e^{-x^2} dx$. (This famous integral appears in the Gaussian Normal distribution.)

   c) Explain why $(\Gamma[1/2])^2 = \int\int dx dy e^{-(x^2+y^2)} = \int_0^\infty dr r \int_0^{2\pi} d\theta e^{-r^2}$, and use this to show that $\Gamma[1/2] = \sqrt{\pi}$. (I evaluated the integral using Mathematica in the KenLec4.nb, here you are asked to verify it for yourself, by directly doing the integral.)

   d) The function $\Gamma[z]$ can be defined by analytic continuation everywhere in the complex z plane. It has poles at (and only at) $z = -n$ for $n = 0, 1, 2, 3, \ldots$. In fact, $\Gamma[z]$ satisfies a nice identity:

$$\Gamma[z]\ \Gamma[1-z] = \frac{\pi}{\sin(\pi z)},$$

   and we saw in class that the right hand side has poles at all integer values of z, with residue $(-1)^n$. Use this identity to find the residues of the poles of $\Gamma[z]$ at $z = 0, -1, -2, -3$. You can check your answers using Mathematica, but should find the answer by hand, directly from the identity. (Alternatively, the poles and residues follow from $\Gamma[z+1] = z\Gamma[z]$.)

   e) Use the result of the previous part, and Cauchy's theorem, to evaluate $\oint_C dz \Gamma[z]$ where C is a circle of radius π in the complex plane, centered at the origin.

2. Consider the differential equation

$$\frac{d^4x}{dt^4} + 2\frac{d^3x}{dt^3} + 9\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 10x = f(t).$$

   (a) Find the general solution of the homogeneous equation (setting $f(t) = 0$) by the method of considering solutions of the form e^{st}. When you do this, you will get a polynomial equation for s, which you should write out explicitly and solve (either factor it by hand or by using Mathematica if needed).

   (b) Verify that you get the same answer via Mathematica, using *DSolve*.

   (c) Suppose that $f(t) = f_0 \cos(\omega t)$. Write down the particular solution, following the discussion in (1.6.44) of Dubin, writing it out explicitly for the present example (either by hand or typing it into Mathematica, but not using Mathematica to calculate).

3. Dubin 1.4.14d.

4. Dubin 1.5.3.

5. Dubin 1.6.3.

Advice: Start all of the exercises early in the week, outside of lab time. The first two questions do not require much mathematica, and the grad students in the tutorial center would be able to help you with them if needed. Use the lab time to get help as needed with debugging your mathematica code for the three exercises from the Dubin book.