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# Quantum Mechanics A (Physics 212A) Fall 2016
# Worksheet 4 – Solutions

## Announcements

- The 212A web site is:

  **http://keni.ucsd.edu/f16/** .

  Please check it regularly! It contains relevant course information!

## Problems

1. **Give it a Kick**

   Consider the $D = 1$ simple harmonic oscillator in its groundstate. Suppose something kicks the system imparting an additional momentum $p_0$. What's the probability the system remains in the ground state?

   (a) What's the new Hamiltonian for the system? Express this in terms of the usual ladder operators $\hat{a}$ and $\hat{a}^\dagger$

   $H_{new} = \frac{(p+p_0)^2}{2m} + \frac{1}{2}m\omega^2 x^2 = H_{old} + \frac{p\ p_0}{m} + \frac{p_0^2}{2m} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \mathbf{i}\frac{p_0}{m}\sqrt{\frac{m\omega}{2}}(\hat{a}^\dagger - \hat{a}) + \frac{p_0^2}{2m}$

   (b) Define a new operator $\hat{A} \equiv \hat{a} - \beta$ where $\beta \equiv \frac{1}{\mathbf{i}\omega}\frac{p_0}{m}\sqrt{\frac{m\omega}{2}}$.

   Show that the $\hat{A}$ are ladder operators: $[\hat{A}, \hat{A}^\dagger] = 1$

   This follows immediately from $[\hat{a}, \hat{a}^\dagger] = 1$ and that $\beta$ is a constant.

   (c) Rewrite the new Hamiltonian in terms of these operators, what do you find?

   $H_{new} = \omega(\hat{A}^\dagger\hat{A} + \frac{1}{2})$

   (d) Relate the original groundstate $|0\rangle$ to the new groundstate $|\beta\rangle$

   Since the new Hamiltonian is another harmonic oscillator it must be that:
   $\hat{A}|\beta\rangle = 0 = (\hat{a} - \beta)|\beta\rangle$ or in other words $\hat{a}|\beta\rangle = \beta|\beta\rangle$ this is a *coherent* state.

   (e) Using $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$ compute $P = |\langle 0|\beta\rangle|^2$

   Hint: Insert identity and use the relation above.

   $|\beta\rangle = \mathbb{1}|\beta\rangle = \sum_n |n\rangle\langle n|\beta\rangle = \sum_n |n\rangle\langle 0|\frac{(\hat{a})^n}{\sqrt{n!}}|\beta\rangle = (\sum_n \frac{\beta^n}{\sqrt{n!}}|n\rangle)\langle 0|\beta\rangle$

   Knowing this consider $\langle\beta|\beta\rangle = 1 = (\sum_n \frac{(|\beta|^2)^n}{n!}\langle n|n\rangle)|\langle 0|\beta\rangle|^2 = e^{|\beta|^2}|\langle 0|\beta\rangle|^2$

   Therefore: $|\langle 0|\beta\rangle|^2 = e^{-|\beta|^2}$

2. **Bogliubov Transformation**

We solved a new Hamiltonian by defining a set of transformed creation/annihilation operators which satisfy the same algebra $[A, A^\dagger] = \mathbb{1}$

More generally consider $\hat{b} = \hat{a}\cosh\eta + \hat{a}^\dagger \sinh\eta$

(a) Show that $[\hat{b}, \hat{b}^\dagger] = \mathbb{1}$

$[\hat{b}, \hat{b}^\dagger] = \cosh^2\eta[a, a^\dagger] - \sinh^2\eta[a, a^\dagger] = 1[a, a^\dagger] = \mathbb{1}$

(b) Show that $\hat{b} = U\hat{a}U^\dagger$ for $U = e^{\frac{\eta}{2}(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger)}$

Time to bust out BCH: $U\hat{a}U^\dagger = \hat{a} + [A, a] + \frac{1}{2}[A, [A, a]] + \cdots$

Where $A \equiv \frac{\eta}{2}(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger)$. Note $[a^\dagger a^\dagger, a] = -2a^\dagger$ and $[aa, a^\dagger] = 2a$

So $[A, a] = \eta a^\dagger$ and $[A, [A, a]] = \eta[A, a^\dagger] = \eta^2 a$

So everything decomposes into odd terms with $a^\dagger$ and even terms $a$. All plus signs. This gives the form of $b$

(c) Show for fermionic operators $\hat{c}^2 = 0 = (\hat{c}^\dagger)^2$ and $\{\hat{c}, \hat{c}^\dagger\} = \mathbb{1}$ that $\hat{d} = \hat{c}\cos\theta + \hat{c}^\dagger \sin\theta$ is the analogous operator

Similar but the anticommutation means $\{d, d^\dagger\} = \cos^2\theta\{\hat{c}, \hat{c}^\dagger\} + \sin^2\theta\{\hat{c}, \hat{c}^\dagger\} = \mathbb{1}$

Now consider the Hamiltonian

$$\hat{H} = \omega\hat{a}^\dagger\hat{a} + \frac{V}{2}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger) \tag{1}$$

(c) Diagonalize the Hamiltonian (1) using the $\hat{b}$ operators for suitably chosen $\eta$

We should look for an operator of the form $H = \Omega b^\dagger b + F$ for some constant $F$

$b^\dagger b = (\cosh^2\eta + \sinh^2\eta)a^\dagger a + \sinh^2\eta + \cosh\eta\sinh\eta(aa + a^\dagger a^\dagger)$

So $\Omega\cosh(2\eta) = \omega$ and $\Omega\sinh(2\eta) = V \implies V = \omega\tanh(2\eta)$

It also must be that $\Omega\sinh^2\eta + F = 0$ ; solving for $\Omega$ and $F$ independently is just algebra. $\Omega = \omega\cosh(2\eta) - V\sinh(2\eta)$

The spectrum is simple now though! $E_n = \Omega n + F$

(d) Show there is a limit on $V$ for which this Hamiltonian makes physical sense

$\Omega$ must be positive. $\omega\cosh(2\eta) > V\sinh(2\eta) \implies V < \frac{\omega}{\tanh(2\eta)} = \frac{\omega^2}{V}$

Thus $V < \omega$