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Quantum Mechanics A (Physics 212A) Fall 2016
Worksheet 3 – Solutions

Announcements

- The 212A web site is:

  http://keni.ucsd.edu/f16/ .

  Please check it regularly! It contains relevant course information!

Problems

1. **Quis Custodiet Ipsos Custodes?** (From Jacobs)

   Projective measurements lead to some weird things.

   Consider a two state system with basis vectors $\{|0\rangle, |1\rangle\}$. We are going to evolve the system according the Hamiltonian $\hat{H} = \frac{\omega}{2}Y$ where Y is the Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

   (a) What is the unitary operator associated with time evolution? Given an initial prepared state of $|\psi_0\rangle = |0\rangle$. Write an expression for $|\psi(t)\rangle$.
   $U = e^{-\mathbf{i}Ht} = e^{-\mathbf{i}\frac{\omega}{2}Yt}$. Recall that $Y|0\rangle = \mathbf{i}|1\rangle$ and that $e^{-\mathbf{i}\frac{\omega}{2}\vec{\sigma}.\hat{n}} = \cos\frac{\omega}{2}\mathbb{1} - \mathbf{i}\vec{\sigma}.\hat{n}\sin\frac{\omega}{2}$
   This implies $|\psi(t)\rangle = \cos\frac{\omega t}{2}|0\rangle + \sin\frac{\omega t}{2}|1\rangle$

   (b) What is the probability, as function of time, to measure $|0\rangle$?
   $P_t[|0\rangle] = \cos^2\frac{\omega t}{2}$

   (c) Suppose we study the system over the time interval $[0, T]$ where $T \gg \delta t \equiv \frac{T}{N}$. We perform a measurement, in this basis, at every time $\frac{T}{N}, \frac{2T}{N}, \cdots$ where N is large. Assuming each measurement is independent from the other, what's the probability that the spin *never* flips to $|1\rangle$?

   Recall that our measurement axiom says we should 'collapse' $|\psi\rangle$ onto the pure state which we measure it to be.

   This implies if we measure $|0\rangle$ at time $t = \frac{T}{N}$ then the time evolution from $\frac{T}{N} \to \frac{2T}{N}$ is the same as if starting from $t = 0$. The probability to not flip is always $\cos^2\frac{\omega T}{2N}$

   The probability for the spin to never flip then is just the product of probabilities to not flip at every measurement.

   $P_{never-flip} = (\cos^2\frac{\omega T}{2N})^N$

(d) Evaluate this probability in the limit of $N \to \infty$.

This is called the *quantum Zeno effect*.

A cheap and dirty way to do this is to take the series expansion at $N = \infty$ and drop all terms in $\mathcal{O}(\frac{1}{N})$. Just replace $\frac{1}{N} \equiv \eta$ and Taylor expand at $\eta = 0$.

$P_{never-flip} \approx 1 - \frac{\omega^2 T^2}{4N}$ which goes quickly to 1. In this limit the spin never flips.

More carefully you should see that it actually *exponentially* approaches 1 as $N \to \infty$; it's pretty dramatic

2. **Building Bloch's Theorem**

Consider a 1D Hamiltonian with a periodic potential $V(x) = V(x + na)$ for $n \in \mathbb{Z}$ and a the lattice spacing.

(a) Define the operator T^n by $T^n|x\rangle = |x + na\rangle$. Show this is a symmetry.

We would need that $[H, T] = 0$ or equivalently $H' = T^\dagger H T = H$. Computing $H'\psi(x) = \langle x|H'|\psi\rangle$. Writing H' in the position basis: $H' = -\frac{1}{2}\partial_{x'}^2 + V(x')$.

$V(x') = V(x + a) = V(x)$ by assumption. One computes the Jacobian: $\frac{\partial x'}{\partial x} = \frac{\partial x + a}{\partial x} = \frac{\partial x}{\partial x} = 1$ to see the kinetic part is also unchanged. Thus H' is invariant and the result is shown!

(b) Assuming H has no shared degeneracy with T, show that any eigenfunctions of this system can be chosen to obey

$$\psi_k(x - a) = e^{-\mathbf{i}ka}\psi_k(x) \tag{1}$$

Recall that $T|k\rangle = e^{-\mathbf{i}ka}|k\rangle$ and $\langle x|k\rangle \equiv \psi_k(x)$.

Since T is a symmetry, and there is no degeneracy, I can diagonalize H in the basis of T: $|k\rangle$. The above implies that $T|k\rangle = e^{-\mathbf{i}ka}|k\rangle$ and that $H|k\rangle = E_k|k\rangle$.

If there were degeneracy then not every eigenvector of H need be an eigenvector of T. This is called *spontaneous symmetry breaking*

The proof then is very easy: $\langle x - a|k\rangle = \langle x|T|k\rangle = e^{-\mathbf{i}ka}\psi_k(x)$

(c) Infer from (1) that one can then write $\psi_k(x) = e^{\mathbf{i}kx}u_k(x)$ where $u_k(x) = u_k(x + a)$

This amounts to showing the claimed form for ψ_k satisfies (1)

$\psi_k(x - a) = e^{\mathbf{i}k(x-a)}u_k(x - a) = e^{-\mathbf{i}ka}e^{\mathbf{i}kx}u_k(x)$ done.

Note that because not every a is a valid symmetry transformation the function u_k is not generically a constant.

Note that k is different from our usual momentum. It's a *crystal momentum*!

(d) Show explicitly that for $P = -\mathbf{i}\partial_x$ that $P\psi_k(x) \neq k\psi_k(x)$

$-\mathbf{i}\partial_x(\psi_k(x)) = ke^{\mathbf{i}kx}u_k(x) - \mathbf{i}e^{\mathbf{i}kx}u_k'(x)$

(e) Show that $\frac{-\pi}{a} \leq k \leq \frac{\pi}{a}$. What is $k + \frac{2\pi}{a}$?

Consider $\psi_{k+\frac{2\pi}{a}}(x) = e^{\mathbf{i}kx}e^{\mathbf{i}\frac{2\pi}{a}x}u_{k+\frac{2\pi}{a}}(x)$ and notice that the product $e^{\mathbf{i}\frac{2\pi}{a}x}u_{k+\frac{2\pi}{a}}(x)$ is still a periodic function of x with period a.

Therefore this is just a relabelling of the function u_k. Call $v_k(x) \equiv e^{\mathbf{i}\frac{2\pi}{a}x} u_{k+\frac{2\pi}{a}}(x)$ Then $\psi_{k+\frac{2\pi}{a}} = e^{\mathbf{i}kx} v_k(x)$; this transformation on k did nothing! Therefore $k + \frac{2\pi}{a} \equiv k$ and that's why we only need the finite region.