University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics A (Physics 212A) Fall 2016
Worksheet 1

Announcements

- The 212A web site is:

  http://keni.ucsd.edu/f16/ .

  Please check it regularly! It contains relevant course information!

Problems

1. **Normal matrices.**

   An operator (or matrix) $\hat{A}$ is *normal* if it satisfies the condition $[\hat{A}, \hat{A}^\dagger] = 0$.

   (a) Show that real symmetric, hermitian, real orthogonal and unitary operators are normal.

   (b) Show that any operator can be written as $\hat{A} = \hat{H} + \mathbf{i}\hat{G}$ where $\hat{H}, \hat{G}$ are Hermitian. [Hint: consider the combinations $\hat{A} + \hat{A}^\dagger, \hat{A} - \hat{A}^\dagger$.] Show that $\hat{A}$ is normal if and only if $[\hat{H}, \hat{G}] = 0$.

   (c) Show that a normal operator $\hat{A}$ admits a spectral representation

   $$\hat{A} = \sum_{i=1}^{N} \lambda_i \hat{P}_i$$

   for a set of projectors $\hat{P}_i$, and complex numbers λ_i.

2. **Gone with a Trace**

   Recall the trace of an operator $\text{Tr}\,[A] = \sum_m \langle m|A|m\rangle$ for the some basis set $\{|m\rangle\}$

   (a) Prove that this definition is independent of basis.
   This implies if A is diagonalizable with eigenvalues λ_i that $\text{Tr}\,[A] = \sum_i \lambda_i$

   (b) Prove the cycle property: $\text{Tr}\,[ABC] = \text{Tr}\,[BCA] = \text{Tr}\,[CAB]$

   (c) Consider an operator A. Show the following identity

   $$\det e^A = e^{\text{Tr}\,[A]} \tag{1}$$

   Hint: Recall that the determinant is the product of eigenvalues

3. **Clock and shift operators.**

Consider an N-dimensional Hilbert space, with orthonormal basis $\{|n\rangle, n = 0, \ldots, N-1\}$. Consider operators $\mathbf{T}$ and $\mathbf{U}$ which act on this N-state system by

$$\mathbf{T}|n\rangle = |n+1\rangle, \quad \mathbf{U}|n\rangle = e^{\frac{2\pi \mathbf{i} n}{N}}|n\rangle \ .$$

In the definition of $\mathbf{T}$, the label on the ket should be understood as its value modulo N, so $N + n \equiv n$ (like a clock).

(a) Find the matrix representations of $\mathbf{T}$ and $\mathbf{U}$ in the basis $\{|n\rangle\}$.

(b) What are the eigenvalues of $\mathbf{U}$? What are the eigenvalues of its adjoint, $\mathbf{U}^\dagger$?

(c) Show that

$$\mathbf{UT} = e^{\frac{2\pi \mathbf{i}}{N}}\mathbf{TU}.$$

(d) From the definition of adjoint, how does $\mathbf{T}^\dagger$ act?

$$\mathbf{T}^\dagger|n\rangle = ?$$

(e) Show that the 'clock operator' $\mathbf{T}$ is normal – that is, commutes with its adjoint – and therefore can be diagonalized by a unitary basis rotation.

(f) Find the eigenvalues and eigenvectors of $\mathbf{T}$.
[Hint: consider states of the form $|\theta\rangle \equiv \sum_n e^{\mathbf{i}n\theta}|n\rangle$.]