10/24/16 Lecture 9 outline

• Last time: position space probability density $\rho(\vec{x},t) = |\psi(\vec{x},t)|^2$ and current $\vec{j}(\vec{x},t) = (\hbar/m)\mathrm{Im}(\psi^*\nabla\psi)$. Note $\int d^3\vec{x}\vec{j} = \langle\vec{p}\rangle/m$. Can also write $\vec{j} = \rho\nabla S/m$, where $\psi \equiv \sqrt{\rho}e^{iS/\hbar}$. E.g. for a plane wave $\nabla S = \vec{p}$. Substituting $\psi \equiv \sqrt{\rho}e^{iS/\hbar}$ into the time dependent SE gives an equation where each S derivative has a $1/\hbar$. In the classical limit we have e.g. $|\nabla S|^2 \gg \hbar|\nabla^2 S|$ and the SE reduces to

$$\frac{1}{2m}|\nabla S|^2 + V(x) + \frac{\partial S(\vec{x},t)}{\partial t} = 0$$

which is the Hamilton-Jacobi equation of classical mechanics with S Hamilton's function. This shows how the SE reduces to classical mechanics in the $S/\hbar \ll 1$ limit. We will soon briefly discuss the path integral description of QM, where S is replaced with the action functional.

For an energy eigenstate, $S = W(x) - Et$.

Example: 3d harmonic oscillator, ground state is $|000\rangle$ and $\psi_{000}(\vec{x}) = c_0^3\psi_0(x)\psi_0(y)\psi_0(z) = c_0^3 e^{-m\omega r^2/2\hbar}$. First excited states are $|100\rangle$, $|010\rangle$, $|001\rangle$, 3-fold degenerate. We'll soon discuss spherically symmetric potentials more generally and see that this degeneracy is related to angular momentum and spherical harmonics, $\ell = 1$, $m = 1, 0, -1$. For now just mention it and illustrate ρ and $\vec{j}$.

• Linear potential, $V = k|x|$. This case does not have a simple solution in terms of a trick – one has to consider the differential equation. Physically it is also less interesting than the SHO. It comes up e.g. for a particle in a constant force field (e.g. a gravitational field close to the earth's surface), if we replace the $x < \infty$ potential with an infinite one. The $|x|$ is not nice near $x = 0$; not physically realistic there. The main reason to mention it is because an arbitrary potential, in the vicinity of its turning point, can be approximated as a linear potential by the first term in the Taylor expansion. The SE can be converted by a change of variables into $u_E''(z) - zu_E(z) = 0$, which is the Airy equation, whose solution is $Ai(z)$. It oscillates for $z < 0$ and has exponential decay for $z > 0$, which is the classically forbidden region. $Ai(z) \to z^{-1/4}(2\sqrt{\pi})^{-1}e^{-2z^{3/2}/3}$ for $z \to \infty$ and $A_i(z) \to |z|^{-1/4}\pi^{-1/2}\cos(2/3|z|^{3/2} - \pi/4)$ for $z \to -\infty$.

The energy levels are determined by the condition that either u_E or u_E' vanishes at $x = 0$ (parity) so E is quantized according to the zeros of Ai or Ai'.