10/19/16 Lecture 7 outline

• Last time: SHO, $H = p^2/2m + \frac{1}{2}m^2\omega^2x^2$. The equation $H|n\rangle = E_n|n\rangle$ in position space becomes a 2nd-order differential equation which has solution given by some special functions. That is a fine way to solve for the E_n and $\langle x|n\rangle$, especially if we like solving differential equations. Happily, there is a much better way to solve this problem, which is simpler, more interesting, and more important than solving a differential equation. It uses creation and annihilation operators

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}(x + ip/m\omega), \qquad \text{so} \qquad a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}(x - ip/m\omega).$$

These satisfy the fundamental property $[a, a^\dagger] = 1$. We can then immediately show that the Hermitian operator $N = a^\dagger a$ has eigenvalues $n = 0, 1, 2\ldots$, and the eigenvectors $|n\rangle$ satisfy $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n\rangle$. Since $H_{SHO} = \hbar(\omega N + \frac{1}{2})$, we're done. If we really want $\langle x|n\rangle$, we can get it from $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$ by replacing $p \to -i\hbar\frac{d}{dx}$ and we can solve for $\psi_0(x) = \langle x|0\rangle$ by using $a|0\rangle = 0$, which in position space becomes a simple first-order differential equation for $\psi_0(x)$:

$$\psi_n(x) = \frac{(m\omega/2\hbar)^{n/2}}{\sqrt{n!}}(x - \frac{\hbar}{m\omega}\frac{d}{dx})^n\psi_0(x), \qquad (x + \frac{\hbar}{m\omega}\frac{d}{dx})\psi_0(x) = 0.$$

We can likewise get $\langle p|n\rangle \equiv \tilde{\psi}_n(p)$, either from inserting $1 = \int dx|x\langle\rangle x|$, which relates it to $\psi_n(x)$ via Fourier transform, or we can directly go to p basis via $\widehat{p} \to p$ and $\widehat{x} \to i\hbar\frac{d}{dp}$. In x space, the groundstate is seen to be a Gaussian centered at $x = 0$: $\psi_0(x) = c_0 e^{-m\omega x^2/2\hbar}$, where c_0 is determined from $\int dx|\psi_0(x)|^2 = 1 = |c_0|^2\sqrt{\pi\hbar/m\omega}$. Let's find the width of the groundstate Gaussian another way: use creation and annihilation operators to show that, in the groundstate $\langle x^2\rangle = \hbar/2m\omega$ and $\langle p^2\rangle = \hbar m\omega/2$, so the uncertainty principle is saturated. Rather than giving the detailed form of $\psi_n(x) = \langle x|n\rangle$, just comment that it has the form $c_n H_n(x\sqrt{m\omega/\hbar})e^{-m\omega x^2/2\hbar}$, where H_n is a polynomial in x of degree n, called a Hermite polynomial ($e^{-t^2+2tx} \equiv \sum_{n=0}^{\infty} H_n(x)t^n/n!$). Notice the qualitative similarity to the particle in the box: bigger n means smaller wavelength, so more nodes of $\psi_n(x)$: the E_n state has n nodes. There are theorems about this in 1d for particles in bounded potentials: the groundstate always has no nodes, and the energy increases with the number of nodes. In this case, $\psi_n(x)$ extends past the classically allowed region, with exponential decay. (For the particle in an infinite box, ψ vanished outside the box only because of $V = \infty$ there, and indeed the solution has an associated discontinuity in ψ' at the ends.)

Note that $P: x \to -x$ and $p \to -p$ which takes $a \to -a$ and $H \to H$ is a symmetry. We see that the state $|n\rangle \to (-1)^n |n\rangle$ under this symmetry. So $\psi_n(x=0) = 0$ for odd n. Also, see that $\langle n|x^r p^s|m\rangle$ is only non-zero if $n+r+s+m$ is even. Actually, using x and p in terms of a and $a^\dagger$, see that this matrix element must have $|n-m| \leq r+s$.

• In the Heisenberg picture we have $\dot{a} = -i\omega a$, hence $a(t) = e^{-i\omega t}a$, where $a = a(0)$. Show that this gives $x(t) = x(0)\cos\omega t + (p(0)/m)\sin\omega t$ and $p(t)$ is then obtained from $p = m\dot{x}(t)$. BCH formula $e^{iB\lambda}Ae^{-iB\lambda} = \sum_{n=0}^{\infty}((i\lambda)^n/n!)[B,[B,\ldots A]]]]]$.

• 2d and 3d particle in a box and SHOs. First discuss general case of $H = H_1 + H_2$ subsystems, where all variables in H_1 commute with those in H_2. Then the Hilbert space is spanned by a tensor product of states from the two subsystems $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, and observables such as E are the sum of those in the two subsystems. In terms of solving the S.E. this is the statement of separation of variables.

• Position space probability density $\rho(\vec{x},t) = |\psi(\vec{x},t)|^2$ and current $\vec{j}(\vec{x},t) = (\hbar/m)\mathrm{Im}(\psi^* \nabla \psi)$. Note $\int d^3\vec{x}\vec{j} = \langle \vec{p}\rangle/m$. Can also write $\vec{j} = \rho \nabla S/m$, where $\psi \equiv \sqrt{\rho}e^{iS/\hbar}$. E.g. for a plane wave $\nabla S = \vec{p}$. Substituting $\psi \equiv \sqrt{\rho}e^{iS/\hbar}$ into the time dependent SE gives an equation where each S derivative has a $1/\hbar$. In the classical limit we have e.g. $|\nabla S|^2 \gg \hbar|\nabla^2 S|$ and the SE reduces to

$$\frac{1}{2m}|\nabla S|^2 + V(x) + \frac{\partial S(\vec{x},t)}{\partial t} = 0$$

which is the Hamilton-Jacobi equation of classical mechanics with S Hamilton's function. This shows how the SE reduces to classical mechanics in the $S/\hbar \ll 1$ limit. We will soon briefly discuss the path integral description of QM, where S is replaced with the action functional.