10/17/16 Lecture 7 outline

• Last time: $i\hbar\partial_t U(t,t_0)=HU(t,t_0)$, $i\hbar|\psi(t)\rangle_S = H|\psi(t)\rangle_S$, and $\mathcal{O}^H=U^\dagger\mathcal{O}U$ has

$$\frac{d}{dt}\mathcal{O}^H=\frac{1}{i\hbar}[\mathcal{O}^H,H]+\frac{\partial}{\partial t}\mathcal{O}^H.$$

• Emphasize that it is first order in t, like Hamilton's equations in phase space. Knowing initial state at $t=0$ fully determines the state (or operators) at later t. No uncertainty or probability here. The uncertainty / probability comes when one *measures* (in a state that isn't en eigenstate).

• SG examples, now with spin precessing in between the measurements. E.g. if there is an external magnetic field in between the SG experiments, it'll make the spin of the state precess via $H=-\vec{\mu}\cdot\vec{B}$ with $\vec{\mu}=ge\vec{S}/2mc$. Take e.g. $\vec{B}=B_0\widehat{z}$ so $H=g|e|S_zB/2mc\equiv\omega S_z$ Then $U=e^{-iHt/\hbar}$ is diagonal in the $|\pm_z\rangle$ basis. Show e.g. that if in the $|+_x\rangle$ state at $t_0=0$, the probability of finding it later in the $|\pm_x\rangle$ state is $\cos^2(\omega t/2)$ and $\sin^2(\omega t/2)$, respectively, and $\langle S_x\rangle=\frac{1}{2}\hbar\cos\omega t$ and $\langle S_y\rangle=\frac{1}{2}\hbar\sin\omega t$ and $\langle S_z\rangle=0$, fitting with the classical picture of precessing in the xy plane with frequency ω. Recall from HW that $e^{i\theta\widehat{n}\cdot\vec{\sigma}}=\cos\theta\mathbf{1}+i\sin\theta\widehat{n}\cdot\vec{\sigma}$. Work out $S_x(t)$ in that basis.

• For a massive particle in a bounding potential, the energy levels are discrete, E_n, with $n=0,1\ldots$. Sometimes there are discrete levels and then a continuum, e.g. for atoms, there are the bound energy levels $E_n<0$, and then a continuum of $E>0$ where the atom is ionized. Consider for the moment the case where there are discrete energy levels E_n, which are the eigenvalues of H, $H|E_n\rangle=E_n|E_n\rangle$. Often just write $|n\rangle$ instead of E_n. The $|E_n\rangle$ form a complete orthonormal basis, so $\langle E_n|E_m\rangle=\delta_{n,m}$ and $1=\sum_n|E_n\rangle\langle E_n|$, and any $|\psi\rangle$ can be thus expanded. The $|E_n\rangle$ in the S-picture time evolve with a simple phase $|E_n(t)\rangle_S=e^{-iE_nt/\hbar}|E_n(t=0)\rangle$, which is physically the same state (quantum states don't depend on the overall normalization), so they are referred to as stationary states. Expanding $|\psi(t)\rangle_S$ in terms of these energy eigenstates reveals how the general state time evolves. E.g. 1d particle in a box.

• Now consider the SHO, $H=p^2/2m+\frac{1}{2}m^2\omega^2x^2$. The equation $H|n\rangle=E_n|n\rangle$ in position space becomes a 2nd-order differential equation which has solution given by some special functions. That is a fine way to solve for the E_n and $\langle x|n\rangle$, especially if we like solving differential equations. Happily, there is a much better way to solve this problem, which is simpler, more interesting, and more important than solving a differential equation. It uses creation and annihilation operators

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}(x + ip/m\omega), \qquad \text{so} \qquad a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}(x - ip/m\omega).$$

These satisfy the fundamental property $[a, a^\dagger] = 1$. We can then immediately show that the Hermitian operator $N = a^\dagger a$ has eigenvalues $n = 0, 1, 2 \ldots$, and the eigenvectors $|n\rangle$ satisfy $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n\rangle$. Since $H_{SHO} = \hbar(\omega N + \frac{1}{2})$, we're done. If we really want $\langle x|n\rangle$, we can get it from $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$ by replacing $p \to -i\hbar\frac{d}{dx}$ and we can solve for $\psi_0(x) = \langle x|0\rangle$ by using $a|0\rangle = 0$, which in position space becomes a simple first-order differential equation for $\psi_0(x)$:

$$\psi_n(x) = \frac{(m\omega/2\hbar)^{n/2}}{\sqrt{n!}}(x - \frac{\hbar}{m\omega}\frac{d}{dx})^n\psi_0(x), \qquad (x + \frac{\hbar}{m\omega}\frac{d}{dx})\psi_0(x) = 0.$$

The solution is a Gaussian centered at $x = 0$: $\psi_0(x) = c_0 e^{-m\omega x^2/2\hbar}$, where c_0 is determined from $\int dx|\psi_0(x)|^2 = 1 = |c_0|^2\sqrt{\pi\hbar/m\omega}$. Let's find the width of the groundstate Gaussian another way: use creation and annihilation operators to show that, in the groundstate $\langle x^2\rangle = \hbar/2m\omega$ and $\langle p^2\rangle = \hbar m\omega/2$, so the uncertainty principle is saturated. Rather than giving the detailed form of $\psi_n(x) = \langle x|n\rangle$, just comment that it has the form $c_n H_n(x\sqrt{m\omega/\hbar})e^{-m\omega x^2/2\hbar}$, where H_n is a polynomial in x of degree n, called a Hermite polynomial ($e^{-t^2+2tx} \equiv \sum_{n=0}^\infty H_n(x)t^n/n!$). Notice the qualitative similarity to the particle in the box: bigger n means smaller wavelength, so more nodes of $\psi_n(x)$: the E_n state has n nodes. There are theorems about this in 1d for particles in bounded potentials: the groundstate always has no nodes, and the energy increases with the number of nodes. Note also that we can take $x \to -x$ and $p \to -p$ which takes $a \to -a$ and $H \to H$. It is a symmetry. We see that the state $|n\rangle \to (-1)^n|n\rangle$ under this symmetry. So $P_n(x = 0) = 0$ for odd n.

Note also that $\psi_n(x)$ extends past the classically allowed region, with exponential decay.