10/10/16 Lecture 5 outline

• Last time: $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$. Use this to prove $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$ for Hermitian A and B. Generalized uncertainty principle: observables whose operators do not commute cannot be simultaneously measured, measuring one messes up measurement of the other. Hence there is a minimal dispersion of their product.

• Position and momentum, and $[x,p] = i\hbar$. So $\Delta x \Delta p \geq \hbar/2$, uncertainty principle. We will see it is minimized for gaussian wave packets at $t = 0$, and we will see that it increases with t. First, use to estimate ground state energy, e.g. particle in a box of size L has $\Delta x \sim L$ and $E \sim p^2/2m \sim (\Delta p)^2/2m \sim \hbar^2/mL^2$ (actual answer is $\pi^2\hbar^2/2mL^2$).

• Position and momentum eigenstates. For positions, $\widehat{x}|x\rangle = x|x\rangle$; generalize to 3d. Momentum as generator of translations. Converting between position and momentum eigenstate bases. Translation generator $U(\vec{a}) = e^{-i\vec{p}\cdot\vec{a}/\hbar}$, satisfies $U(\vec{a})|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$, so $\langle\vec{x}|\psi\rangle = \psi(\vec{x})$ and $\langle\vec{x}|U|\psi\rangle = \psi(\vec{x}-\vec{a})$. Checks: $U(\vec{a})^{-1} = U(-\vec{a})$. Such exponentials frequently appear because tiny transformations are combined by exponentiation, since $\lim_{N\to\infty}(1+A/N)^N = e^A$. Note translations are commutative $U(\vec{a})U(\vec{b}) = U(\vec{a}+\vec{b}) = U(\vec{b})U(\vec{a})$, so $[p_i, p_j] = 0$. As we'll discuss later, rotations are generated by $U(\vec{\phi}) = e^{-i\vec{\phi}\cdot\vec{J}}$, where $\vec{J}$ is angular momentum, and the fact that 3d rotations do not commute, $U(\vec{\phi}_1)U(\vec{\phi}_2) \neq U(\vec{\phi}_2)U(\vec{\phi}_1)$, implies that $[J_a, J_b] = i\hbar\epsilon_{abc}J_c$, as we saw for the S_a.

• Consider first 1d case. $\langle x|e^{-i\widehat{p}a/\hbar}|\psi\rangle = \psi(x-a) = e^{-a\frac{d}{dx}}\psi(x)$, where the last one is Taylor's series. So we can write $\widehat{p} = -i\hbar\frac{d}{dx}$ in the $\langle x|$ basis. Then $\langle x|\widehat{p}\rangle p$ gives $-i\hbar\frac{d}{dx}\psi_p(x) = p\psi_p(x)$, where $\psi_p(x) \equiv \langle x|p\rangle = Ne^{ipx/\hbar}/\sqrt{2\pi\hbar}$, where it is common to take $N = 1/\sqrt{2\pi\hbar}$ or $N = 1$, but that is just a convention.

• The position and momentum eigenstates are delta-function normalized: $\langle x'|x\rangle = \delta(x-x')$ and $\langle p'|p\rangle = N'\delta(p-p')$, where it is common to take $N' = 1$ or $N' = 2\pi\hbar$. Generalization to 2d or 3d.

• Recall $P(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\langle x\rangle)^2/2\sigma^2}$, so take $\psi(x) = e^{ikx}\sqrt{P(x)}$. Find $\psi(p)$ is also a Gaussian, centered at $\langle p\rangle = \hbar k$, and minimizes the uncertainty principle.