10/3/16 Lecture 3 outline

• N state Hilbert space. Basis vectors. Operators A and B and their eigenbases. Consider $[A, B] = 0$ and $[A, B] \neq 0$. Unitary transformation between bases. Recall rotation between bases for vectors and compare with bra ket inner product and unitary transformations.

• Last time: two state system, e.g. happy cat or sad cat. Can only measure one or the other, mutually exclusive, and a general state is a linear superposition with complex coefficients. Physical observables are Hermitian operators. Can work in basis of their eigenstates. We started to discuss $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ in the $|\pm_{\hat{z}}\rangle$ basis. Note that $\vec{S}^2 = (3/4)\hbar^2\mathbf{1}$, and $[S^a, S^b] = i\hbar\epsilon^{abc}S^c$, which are basis independent. We wrote $|\pm_{\hat{x}}\rangle$ and $|\pm_{\hat{y}}\rangle$ in the $|\pm_{\hat{z}}\rangle$ basis.

• Note that $|\pm_{\hat{x}}\rangle$ and $|\pm_{\hat{y}}\rangle$ are related to $|\pm_z\rangle$ by unitarity transformations. Aspects of unitary transformations and change of bases. Indeed, these transformations are examples of rotations. $U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{S}/\hbar}$, e.g. $U(2\pi) = -1$. Rotate $\pi/2$ around y axis to rotate z into x eigenstates. More generally, observables A and UAU^{-1} are unitary equivalent.

ended here

• It follows from $[S_a, S_b] \neq 0$ for $a \neq b$ that spin along different axes cannot be simultaneously diagnonalized, and hence they cannot be simultaneously measured.

• Schwarz inequality: $|||\chi\rangle||^2 = \langle\chi|\chi\rangle \geq 0$. Apply to $|\chi\rangle = |\alpha\rangle + x|\beta\rangle$ and minimize in x, taking it to be $-\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle$, find $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$. Use this to prove $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}\langle[A, B\rangle|^2$ for Hermitian A and B. Use $\Delta A\Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}$ and note that the first term is anti-Hermitian and the second is Hermitian, so their expectation values are pure imaginary and real, respectively.

• Compute dispersion of $S_{x,y,z}$ in $|+_{\hat{z}}\rangle$ state. Check above inequality for $A = S_a$ and $B = S_b$, so $[A, B] = i\hbar C$, with $C = S_c$ and $\epsilon_{abc} = 1$.

• Stern Gerlach again, using bras and kets.

• Position and momentum, and $[x, p] = i\hbar$. So $\Delta x\Delta p \geq \hbar/2$, uncertainty principle. Can use to estimate ground state energy, e.g. particle in a box of size L has $\Delta x \sim L$ and $E \sim p^2/2m \sim (\Delta p)^2/2m \sim \hbar^2/mL^2$. Actual ground state energy is $\pi^2\hbar^2/2mL^2$.

• Position and momentum eigenstates. Momentum as generator of translations. Converting between position and momentum eigenstate bases. Translation generator $U(\vec{a}) = e^{-i\vec{p}\cdot\vec{a}/\hbar}$, satisfies $U(\vec{a})|\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$, so $\langle\vec{x}|\psi\rangle = \psi(\vec{x})$ and $\langle\vec{x}|U|\psi\rangle = \psi(\vec{x} - \vec{a})$.

Checks: $U(\vec{a})^{-1} = U(-\vec{a})$. Such exponentials frequently appear because tiny transformations are combined by exponentiation, since $\lim_{N\to\infty}(1+x/N)^N = e^x$.

• Consider first 1d case. $\langle x|e^{-i\widehat{pa}/\hbar}|\psi\rangle = \psi(x-a) = e^{-a\frac{d}{dx}}\psi(x)$, where the last one is Taylor's series. So we can write $\widehat{p} = -i\hbar\frac{d}{dx}$ in the $\langle x|$ basis. Then $\langle x|\widehat{p}\rangle p$ gives $-i\hbar\frac{d}{dx}\psi_p(x) = p\psi_p(x)$, where $\psi_p(x) \equiv \langle x|p\rangle = Ne^{ipx/\hbar}/\sqrt{2\pi\hbar}$, where it is common to take $N = 1/\sqrt{2\pi\hbar}$ or $N = 1$, but that is just a convention.

• The position and momentum eigenstates are delta-function normalized: $\langle x'|x\rangle = \delta(x-x')$ and $\langle p'|p\rangle = N'\delta(p-p')$, where it is common to take $N' = 1$ or $N' = 2\pi\hbar$.

• Generalization to 2d or 3d.

• Momentum generates spatial translations, and the Hamiltonian (energy) generates time translations. Schrodinger equation: $i\hbar\partial_t U(t,t_0) = HU(t,t_0)$. For time independent H, this is simple to integrate: $U = e^{-iH(t-t_0)/\hbar}$. For t dependent H, one gets instead (Dyson) $U(t,t_0) = T\exp(-(i/\hbar)\int_{t_0}^{t} dt' H(t'))$, where T denotes time ordering. We won't need that here.