11/30/16 Lecture 19 outline

• Last time: spherical tensor operators: let $T_q^{(k)}$ be an operator with $\ell = k$ and $m = q$. For example, $T_0^{(2)} = U_+V - +2U_0V_0 + U_-U_+$ where $\vec{U}$ and $\vec{V}$ are two vectors and $U\pm = \mp(U_x \pm iU_y)/\sqrt{2}$ and $U_0 = U_z$. They have $[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$ and $[J_\pm, T_q^{(k)}] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}T_{q\pm1}^{(k)}$, i.e. $T_q^{(k)}$ transform like $|k, q\rangle$.

• Wigner-Eckart theorem:

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = (2j+1)^{-1/2}\langle jk; mq|jk; j'm'\rangle\langle \alpha' j'||T^{(k)}||\alpha j\rangle.$$

The idea is that the product $T_q^{(k)}|\alpha, jm\rangle$ is adding the angular momentum of $T_q^{(k)}$ to that of $|\alpha, jm\rangle$ so the inner product with $\langle \alpha', j'm'|$ is given by a Clebsch Gordon coefficient. The first term on the RHS is a CG coefficient, which is zero unless $m' = q + m$ and $|j-k| \leq j' \leq j+k$. The last term is independent of m and m'; this is where the symmetry gives some helpful mileage. Example: for a scalar operator S get

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = (2j+1)^{-1/2}\delta_{j'j}\delta_{m'm}\langle \alpha' j'||T^{(k)}||\alpha j\rangle.$$

For a vector operator $\vec{V}$ get $j' - j = 0, \pm 1$ and $m' - m = \pm 1, 0$. Useful in perturbation theory for radiation (212b and 212c). Another immediate application: a state of spin j cannot have a non-zero expectation value of an operator with angular momentum ℓ unless $\ell \leq 2j$. Therefore, a particle of spin zero cannot have a non-zero magnetic dipole moment and a particle with spin 1/2 cannot have an electric quadrupole moment.

• Density operators and pure versus mixed ensembles. Up to now, we have been discussing pure quantum states. In such a state, the expectation value of any operator is $\langle \mathcal{O}\rangle \equiv \langle\psi|\mathcal{O}|\psi\rangle$. Now consider a system with some additional uncertainty. For example, an electron that is fresh out of the oven, which has not yet had its spin measured by a Stern-Gerlach experiment. We say it is unpolarized, meaning that its polarization is random and if one considers an ensemble then some will have one polarization some will have another. This is a mixed state. For a mixed state, have $\langle\langle\mathcal{O}\rangle\rangle = \mathrm{tr}\rho\mathcal{O}$, where $\rho = \sum_i p_i|i\rangle\langle i|$, where $\mathrm{tr}\rho = 1$ is a normalization condition. For a pure state, $p_i = \delta_{i,i_0}$. Note $\mathrm{tr}\rho^2 \leq 1$, where the inequality is saturated iff it is a pure state.

In the Schrodinger picture, find $i\hbar\frac{d\rho}{dt} = [H, \rho]$, like the Heisenberg equations of motion but with opposite sign. Indeed, in classical mechanics + quantizing we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}_{PB} \to \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i\hbar}[F, H],$$

so the density operator has $\frac{d\rho}{dt}$, which is the quantum version of Liouville's theorem about the phase space distribution behaving as an incompressible fluid.

Examples from spin 1/2 with pure states and mixed states: $\rho \to \frac{1}{2}(1+\hat{n}\cdot\vec{\sigma})$ for a pure state, where $\frac{1}{2}\hbar\hat{n} = \langle\vec{S}\rangle$. For a mixed state, $\rho \to \frac{1}{2}(1+(p_a-p_b)\hat{n}\cdot\vec{\sigma})$ and $\langle\vec{S}\rangle = \frac{1}{2}\hbar(p_a-p_b)\hat{n}$ and p_a and p_b are probabilities with $p_a + p_b = 1$.

Aside: von Neumann entropy is $S = -k_B \mathrm{tr}\rho \ln \rho$; it is zero for a pure state.

• Density matrices for subsystems if we cannot access the full system. See e.g. McGreevy's lectures, p. 115: suppose that we have two spin $\frac{1}{2}$ electrons in a state with total spin 0: $|\mathrm{Bohm}\rangle = (|+-\rangle - |-+\rangle)/\sqrt{2}$. We can get a density matrix from $\rho_A = \mathrm{tr}_B|\mathrm{Bohm}\rangle\langle\mathrm{Bohm}| = \frac{1}{2}\mathbf{1}_A$, which is a maximally impure state.

• Spin correlation measurements and Bell's inequality. EPR thought experiment. If Alice measures $\vec{S}\cdot\hat{a} = \pm\hbar/2$, then Bob will measure $\vec{S}\cdot\hat{a} = \mp\hbar/2$ with 100% probability, whereas if Alice does not do the measurement then Bob would get either possibility half the time. This rightly bothered Einstein, Podolsky, and Rosen because the experiments could be spacelike separated. Now if Alice measures $\vec{S}\cdot\hat{a} = +\hbar/2$ and Bob measures $\vec{S}\cdot\hat{b}$, the result according to QM will be $+\hbar/2$ with $P(\hat{a}+;\hat{b}+) = \frac{1}{2}\sin^2(\theta_{ab}/2)$ where $\frac{1}{2}$ is the probability of the initial experiment giving $\vec{S}_1\cdot\hat{a}$ being $+$.

If there were hidden variables and classical physics, then would find

$$P(\hat{a}+;\hat{b}+) \le P(\hat{a}+;\hat{c}+) + P(\hat{c}+;\hat{b}+).$$

This is Bell's inequality, which is a simple statement about probabilities if three things are measured, each of which has two options. Then e.g. $P(1+,2+) = (N_3+N_4)/N_T$ where $N_T = \sum_{i=1}^{8} N_i$ and $P(1+,3+) = (N_2+N_4)/N_T$ and $P(2+,3+) = (N_3+N_7)/N_T$ and the above inequality is simply $N_3+N_4 \le (N_2+N_4)+(N_3+N_7)$.

In QM on the other hand, this inequality is generally violated:

$$\sin^2(\tfrac{1}{2}\theta_{ab}) \le \sin^2(\tfrac{1}{2}\theta ac) + \sin^2(\tfrac{1}{2}\theta_{cb})???.$$

Taking e.g. $\theta_{ac} = \theta_{cb} = \theta$ and $\theta_{ab} = 2\theta$, this inequality is violated for $0 < \theta < \pi/2$. QM says that Bell's inequality can be violated. Experiment agrees with QM (Aspect).