11/28/16 Lecture 18 outline

• Last time: $|j_1 j_2; m_1 m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$, can give $|j_1 j_2; jm\rangle$ for $|j_1 - j_2| \leq j \leq j_1 + j_2$, with j mod integers (i.e. if $j_1 + j_2$ is an integer, then so are all j, and if it is half-integer then so are all j). The Clebsch-Gordon coefficients $\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle$. Using $J_z = J_{1z} + J_{2z}$, show $m = m_1 + m_2$. Recipe: start with case $j = j_1 + j_2$, $m = j_1 + j_2$, where the only possibility is $|j_1 j_1\rangle \otimes |j_2 j_2\rangle$. Now use $J_- = J_{1-} + J_{2-}$ to get all m values for $j = j_1 + j_2$. Now get $j = m = j_1 + j_2 - 1$ by orthogonality:

$$|j_1 + j_2; j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}} |j_1, j_1 - 1\rangle |j_2 j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}} |j_1 j_1\rangle |j_2 j_2 - 1\rangle,$$

$$|j_1 + j_2 - 1; j_1 + j_2 - 1\rangle = -\sqrt{\frac{j_2}{j_1 + j_2}} |j_1, j_1 - 1\rangle |j_2 j_2\rangle + \sqrt{\frac{j_1}{j_1 + j_2}} |j_1 j_1\rangle |j_2 j_2 - 1\rangle.$$

Now lower m to get all $j = j_1 + j_2 - 1$ cases. Now get $j = m j_1 + j_2 - 2$, by imposing orthogonality with the known (from previous steps) vectors with $j_1 + j_2$ and $j_1 + j_2 - 1$ and $m = j_1 + j_2 - 2$. Keep going until done.

• Example of combining spin 1 and spin 1/2. Clebsch Gordon coefficients.

• Example of combining three spin 1/2s.

• Symmetry or antisymmetry for identical Bosons or Fermions.

• Consider the case $j_1 = \ell$ an integer, and $j_2 = \frac{1}{2}$. This is of use for Hydrogen etc where the electron has both orbital and spin angular momentum. Get $j = \ell \pm \frac{1}{2}$ for $\ell > 0$, and $j = \frac{1}{2}$ for $\ell = 0$. Note that $\vec{L} \cdot \vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$ is $(\hbar^2/2)(j(j+1) - \ell(\ell+1) - 3/4)$ is $\ell\hbar^2/2$ for $j = \ell + \frac{1}{2}$ and $-(\ell+1)\hbar^2/2$ for $j = \ell - \frac{1}{2}$.

• Atomic notation: ${}^{2S+1}L_J$, with $L = 0, 1, 2, 3, \ldots$ denoted by $S, P, D, F, \ldots$ e.g. ${}^2P_{3/2}$ means $\ell = 1$, $s = 1/2$, $j = 3/2$. The ground state of He is ${}^1S_0$.

• Recall $[J_a, J_b] = i\hbar\epsilon_{abc} J_c$. Likewise for any vector V_a, have $[V_a, J_b] = i\hbar\epsilon_{abc} V_c$; this is determined by the fact that angular momentum generates rotations. Can be easily verified for $V_a = X_a$, or $V_a = P_a$ as examples.

Define $T_0^{(1)} \equiv \sqrt{\frac{3}{4\pi}} V_z$ and $T_{\pm 1}^{(1)} = \sqrt{\frac{3}{4\pi}} (\mp \frac{V_x \pm i V_y}{\sqrt{2}})$. These definitions fit with $Y_m^{\ell=1}$ in terms of (θ, ϕ) if we take $\vec{V} = \vec{X}/r$. With these definitions, we have $[J_z, T_m^{(1)}] = \hbar m T_m^{(1)}$ and $[J_\pm, T_m^{(1)}] = \hbar\sqrt{(1 \pm m)(1 \pm m + 1)} T_{m\pm 1}^{(1)}$. Recall $J_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar |j, m \pm 1\rangle$. The spherical vector transforms the same way. More generally spherical tensor operators: let $T_q^{(k)}$ be an operator with $\ell = k$ and $m = q$. For example, $T_0^{(2)} = U_+ V - +2U_0 V_0 + U_- U_+$ where $\vec{U}$ and $\vec{V}$ are two vectors and $U\pm = \mp(U_x \pm i U_y)/\sqrt{2}$

and $U_0 = U_z$. They have $[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$ and $[J_\pm, T_q^{(k)}] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}T_{q\pm 1}^{(k)}$, i.e. $T_q^{(k)}$ transform like $|k, q\rangle$.

- Wigner-Eckart theorem:

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = (2j+1)^{-1/2}\langle jk; mq|jk; j'm'\rangle\langle \alpha' j'||T^{(k)}||\alpha j\rangle.$$

The first term is a CG coefficient, which is zero unless $m' = q + m$ and $|j - k| \leq j' \leq j + k$. The last term is independent of m and m'; this is where the symmetry gives some helpful mileage.

Example: for a scalar operator S get

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = (2j+1)^{-1/2}\delta_{j'j}\delta_{m'm}\langle \alpha' j'||T^{(k)}||\alpha j\rangle.$$

For a vector operator $\vec{V}$ get $j' - j = 0, \pm 1$ and $m' - m = \pm 1, 0$. Useful in perturbation theory for radiation.