11/16/16 Lecture 15 outline

• Last time: $U = e^{-i\vec{\alpha}\cdot\vec{J}/\hbar}$ a unitary operator, in representation of the group $SU(2)$. The group multiplication law is ensured by

$$[J_a, J_b] = i\hbar\epsilon_{abc}J_c.$$

We can find simultaneous eigenstates of J_z and $\vec{J}^2$, since $[\vec{J}^2, J_a] = 0$: write $|a, b\rangle$ as an eigenstate of $\vec{J}^2$ and J_z with eigenvalues a, b. In such an eigenstate, it follows from the above commutator that J_x and J_y will have some uncertainty if $b \neq 0$. Define $J_\pm \equiv (J_z \pm iJ_y)$, which satisfy $[J_z, J_\pm] = \pm\hbar J_\pm$ and $[J_+, J_-] = 2\hbar J_z$. Write these out explicitly for $\vec{J} \to \frac{1}{2}\hbar\vec{\sigma}$. If $|a, b\rangle$ is an eigenstate of $\vec{J}^2$ and J_z with eigenvalues a, b, then $J_\pm|a, b\rangle = c_\pm|a, b\pm\hbar\rangle$. The proportionality constant $c_\pm$ will be determined from $J_\mp J_\pm = \vec{J}^2 - J_z^2 \mp \hbar J_z$. Note first that $\vec{J}^2 - J_z^2 = \frac{1}{2}(J_+J_+^\dagger + JJ_+^\dagger J_+)$ must have non-negative expectation values in any state, and therefore $a \geq b^2$. So we can't raise and lower b beyond $\sqrt{a}$; there is a b_{min} and a b_{max}, with $J_-|a, b_{min}\rangle = 0$ and $J_+|a, b_{max}\rangle = 0$. Obtain $a = b_{max}(b_{max} + \hbar) = b_{min}(b_{min} - \hbar)$, so $b_{min} = -b_{max}$. Finally, can raise / lower one to the other, so $b_{max} - b_{min} = 2b_{max} = n\hbar$ for some integer n. There are then $n + 1$ values of b; e.g. for $\vec{J} \to \frac{1}{2}\hbar\vec{\sigma}$ we have $n = 1$ and this is a 2d representation. Write $j = n/2$ and then $a = \hbar^2 j(j + 1)$, and $b = m\hbar$ with $m = -j, -j + 1, \ldots, j$.

Considering matrix elements in $|jm\rangle$ states, which are orthonormal,

$$J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar|j, m \pm 1\rangle.$$

• The rotation group elements are, in the j representation, given by $(2j+1)\times(2j+1)$ matrices:

$$\mathcal{D}^{(j)}_{m'm} = \langle j, m'|\exp(-i\vec{J}\cdot\vec{\phi}/\hbar)|j, m\rangle.$$

• Orbital angular momentum is a special case, where ℓ must be integral. Writing $\vec{L} = \vec{x} \times \vec{p}$ and going to $|\vec{x}\rangle$ basis, the $\vec{L}$ are angular derivatives: $L_z \to -i\hbar\partial_\phi$, $L_\pm \to -i\hbar e^{\pm i\phi}(\pm i\partial_\theta - \cot\theta\partial_\phi)$. And $\vec{L}^2$ is related to the angular part of the Laplacian in spherical coordinates. This is the same as how, in classical mechanics, we replace $\vec{p}^2 \to p_r^2 + \vec{L}^2/r^2$. $\langle\vec{x}|L^2|\psi\rangle = -\hbar^2(\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta))\psi(\vec{x})$.

The spherical harmonics are $Y_\ell^m(\theta, \phi) = \langle\theta\phi|\ell, m\rangle$. E.g. their orthonormality conditions are obtained by $\delta_{\ell\ell'}\delta_{mm'} = \langle\ell'm'|\ell m\rangle = \int d\Omega\langle\ell'm'|\theta\phi\rangle\langle\theta\phi|\ell m\rangle$, where $d\Omega = d\phi d(\cos\theta)$ is the solid angle element.

• Spherically symmetric $V(r)$ means that $[H, L_a] = 0$, so we can find simultaneous eigenstates $|E, \ell, m\rangle$. Writing $\vec{p}^2 \to p_r^2 + L^2/r^2$, we find that $\vec{x}|E\ell m\rangle = R_{E,\ell}(r)Y_{\ell,m}(\theta, \phi)$, where $R_{E,\ell}$ satisfies the radial SE

$$\left(-\frac{\hbar^2}{2mr^2}\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{\ell(\ell+1)\hbar^2}{2mr^2} + V(r)\right) R_{E,\ell}(r) = ER_{E,\ell}(r).$$

It looks a little nicer for $u_{E,\ell}(r) \equiv rR_{E,\ell}(r)$:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_{eff}(r)u = Eu, \qquad V_{eff} = V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}.$$

If we assume that $r^2V(r) \to 0$ for $r \to 0$, then the angular momentum barrier wins and the SE implies $u(r) \to Ar^{\ell+1} + Br^{-\ell}$ in this limit, and the condition that $j_r = \hat{r} \cdot \vec{j} = \frac{\hbar}{m} Im(\psi^* \partial_r \psi) \to 0$ for $r \to 0$ excludes the second term, so $R_{E,\ell} \to r^\ell$ for $r \to 0$. So wavefunction vanishes at origin except for $\ell = 0$; this is the angular momentum barrier.