11/14/16 Lecture 14 outline

• Recall $U = \exp(-iHt/\hbar)$ is the time evolution operator, and $T(\vec{a}) = e^{-i\vec{p}\cdot\vec{a}}$ is the translation generator. Recall that they are unitary, with Hermitian operators in the exponent, and why we get exponentials from combining lots of infinitesimal transformations.

Recall from classical mechanics that angular momentum is conjugate to angles; angular momentum is the generator of rotations.

$$R(\vec{\phi}) = e^{-i\vec{\phi}\cdot\vec{J}/\hbar}.$$

• Example: consider a particle in the plane. The position eigenkets $|x\rangle|y\rangle$ can be written in polar coordinates. Then $L_z = (\vec{x}\times\vec{p})_z$ can be written in position basis as $L_z = -i\hbar\frac{\partial}{\partial\phi}$ and $R(\phi_0) = e^{-\phi_0\partial_\phi}$ has $\langle\phi|R(\phi_0)|\psi\rangle = \psi(\phi-\phi_0)$ where $\langle\phi|\psi\rangle = \psi(\phi)$. Counting of rotations in 3d: 3 independent angles, e.g. Euler angles. Rotation $\vec{\alpha} = \alpha\widehat{n}$.

• Rotations form a symmetry group, with a product rule: combining any two rotations gives another rotation. This group has a name: $SO(3)$ or $SU(2)$. The difference is that $SU(2)$ allows for the possibility that a 2π rotation changes the wavefunction by -1. This is what happens for spinors, which we refer to as half-integral spin, and a theorem (spin-statistics) says that the spin is half-integral iff the object is a Fermion, and integral iff it is a boson. For integral spin / bosons, it doesn't matter if we call the rotation group $SO(3)$ or $SU(2)$. For half-integral spins / Fermions we have to call the rotation group $SU(2)$. In any event, the group product rule corresponds, in quantum mechanics, to commutation relations:

$$[J_a, J_b] = i\hbar\epsilon_{abc}J_c.$$

We met these already for the 2-state spin-half, where $\vec{J}\to\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ satisfies this. And we studied there expressions like the above $R(\vec{\phi})$, written as 2×2 matrices.

• It follows from the above that $[\vec{J}^2, J_a] = 0$, so we can find simultaneous eigenstates of $\vec{J}^2$ and J_z. Define $J_\pm \equiv (J_z \pm iJ_y)$, which satisfy $[J_z, J_\pm] = \pm\hbar J_\pm$ and $[J_+, J_-] = 2\hbar J_z$. Write these out explicitly for $\vec{J}\to\frac{1}{2}\hbar\vec{\sigma}$. If $|a,b\rangle$ is an eigenstate of $\vec{J}^2$ and J_z with eigenvalues a, b, then $J_\pm|a,b\rangle = c_\pm|a, b\pm\hbar\rangle$. The proportionality constant $c_\pm$ will be determined from $J_\mp J_\pm = \vec{J}^2 - J_z^2 \mp \hbar J_z$. Note first that $\vec{J}^2 - J_z^2 = \frac{1}{2}(J_+J_+^\dagger + JJ_+^\dagger J_+)$ must have non-negative expectation values in any state, and therefore $a \geq b^2$. So we can't raise and lower b beyond $\sqrt{a}$; there is a b_{min} and a b_{max}, with $J_-|a, b_{min}\rangle = 0$ and $J_+|a, b_{max}\rangle = 0$. Obtain $a = b_{max}(b_{max}+\hbar) = b_{min}(b_{min}-\hbar)$, so $b_{min} = -b_{max}$. Finally, can raise / lower one to the other, so $b_{max} - b_{min} = 2b_{max} = n\hbar$ for some integer n. There are then $n+1$ values of b; e.g. for $\vec{J}\to\frac{1}{2}\hbar\vec{\sigma}$ we have $n = 1$ and this is a 2d representation. Write $j = n/2$ and then $a = \hbar^2 j(j+1)$, and $b = m\hbar$ with $m = -j, -j+1, \ldots, j$.