11/7/16 Lecture 12 outline

• Recall propagator: $K(x_2, t_2; x_1, t_1) \equiv \langle x_2|U(t_2, t_1)|x_1\rangle$. Evaluate by inserting complete set of energy eigenstates. E.g. for free particle:

$$K_{free} = \int \frac{dp}{2\pi\hbar} \exp[i(p(x_2 - x_1) - p^2(t_2 - t_1)/2m)/\hbar] =$$

$$= \sqrt{\frac{m}{2\pi i\hbar(t_2 - t_1)}} \exp[im(x_2 - x_1)^2/2\hbar(t_2 - t_1)].$$

For the SHO get

$$K_{SHO} = \sum_n u_n(x_2)u_n^*(x_1)e^{-iE_n(t_2-t_1)/\hbar} =$$

$$\sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega(t_2 - t_1))}} \exp[im\omega \left((x_2^2 + x_1^2)\cos\omega(t_2 - t_1) - 2x_2x_1\right)/2\hbar \sin(\omega(t_2 - t_1))].$$

These look a bit disgusting but are actually nice: the exponentials are the expected Hamilton functions from classical mechanics, fitting with our discussion before. The fact that they are precisely the classical result, without additional quantum corrections, is special to cases where every term in the Hamiltonian is at most quadratic. In terms of the path integral, the WKB approximation is related to a saddle point approximation of integrals, and the integrals reduce to Gaussians for the case of quadratic actions, and the saddle point approximation in such special cases happens to be exact.

E.g. for a free particle we can evaluate $S[x_{cl}, \dot{x}_{cl}] = \int_{t_1,x_1}^{t_2,x_2} dt \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(x_2 - x_1)^2/(t_2 - t_1)$. For a SHO, $S[x_{cl}, \dot{x}_{cl}] = \int dt(\frac{1}{2}A^2m\omega^2)(\sin^2(\omega t + \phi) - \cos^2(\omega t + \phi)) = \ldots$ where we eliminate A and ϕ in terms of (x_1, t_1) and (x_2, t_2). Some interesting general properties of S_{cl}:

$$\frac{\partial S_{cl}}{\partial t_2} = -E, \qquad \frac{\partial S_{cl}}{\partial x_2} = p.$$

We will use these soon.

• Note that it follows from the above definition that

$$K(x_3, t_3; x_1, t_1) = \int dx_2 K(x_3, t_3; x_2, t_2)K(x_2, t_2; x_1, t_1),$$

$$\psi(x_2, t_2) = \int dx_1 K(x_2, t_2; x_1, t_1)\psi(x_1, t_1).$$

The Kernal K depends on the theory, but not the initial state condition. The wavefunction $\psi(x, t)$ depends on the initial state.

• Dirac commented that $\psi \sim e^{iS/\hbar}$ where S is the action. Feynman then showed how to get a very interesting formulation of QM using the propagator. Get

$$K(x_2, t_2; x_1, t_1) = \int [dx(t)] e^{iS[x(t)]/\hbar},$$

where the integral is over all paths that start at (x_1, t_1) and end at (x_2, t_2). Conceptually it is very interesting. Picture of interference through a double slit, multiple slits, and extrapolating to filling space with infinitely many, infinitely fine, pretend slits – that's the path integral. It generalizes beautifully to quantum field theory. But is sometimes not the simplest way to solve non-relativistic QM problems, as seen for the free particle and SHO. One benefit is that the integral over paths can be discretized and put on a computer. This is what lattice gauge theorists (e.g. Julius Kuti) do in the context of quantum field theory, e.g. to understand the theory behind the strong nuclear force. The classical limit: if $S \gg \hbar$, the rapidly oscillating integral is sharply peaked around the classical path, since the classical EOM extremizes S.

• Free particle example, take $x_0 \equiv x_i$ and $x_{N+1} \equiv x_f$.

$$K(x_f, t_f; x_i, t_i) = \left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2} \int \prod_{i=1}^{N} dx_i \exp[\frac{im}{2\hbar\epsilon} \sum_{i=1}^{N+1} (x_i - x_{i-1})^2]$$

Where we take $\epsilon \to 0$ and $N \to \infty$, with $N\epsilon = T$ held fixed. Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation):

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped.

More generally, use Gaussian integrals:

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(-A_{ij}\phi_i\phi_i + B_i\phi_i) = \pi^{N/2}(\det A)^{-1/2} \exp(A^{-1}_{ij} B_i B_j/4).$$

After $n-1$ steps, get integral:

$$\left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2} \exp[\frac{m}{2\pi i\hbar n\epsilon}(x_n - x_0)^2].$$

So the final answer for the free particle

$$K(x_f, t_f; x_i; t_i) = \left(\frac{2\pi i\hbar T}{m}\right)^{-1/2} \exp[im(x_b - x_a)^2/2\hbar T].$$

which agrees with the answer that we obtained (via just one dp Gaussian integral in the usual formulation of QM).